IMP

$$\frac{\partial (g \circ f)_j}{\partial x_i}(x) = \sum_{l=1}^{m} \left. \frac{\partial g_j}{\partial y_l} \right|_{y=f(x)} \cdot \frac{\partial f_l}{\partial x_i}(x)$$

Ex) $g(x,y) = f(\overbrace{x+y}^{u}, \overbrace{x^2-y^2}^{v})$

$$\frac{\partial g}{\partial x}(x,y) = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial x} = \frac{\partial f}{\partial u} \cdot 1 + \frac{\partial f}{\partial v} \cdot 2x = \frac{\partial f}{\partial u} + 2x \frac{\partial f}{\partial v}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial g}{\partial x}\right)(x,y) = \frac{\partial}{\partial x}\frac{\partial f}{\partial u} \cdot 1 + 2\frac{\partial f}{\partial v} + 2x \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial v}\right)$$

$$= \frac{\partial}{\partial u}\frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\frac{\partial f}{\partial u} \cdot \frac{\partial v}{\partial x} + \frac{2\partial f}{\partial v} + 2x\left(\frac{\partial}{\partial v}\frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial x} + \frac{\partial}{\partial u}\frac{\partial f}{\partial v} \cdot \frac{\partial u}{\partial x}\right)$$

$$= \frac{\partial^2 f}{\partial u^2} + 2x \frac{\partial^2 f}{\partial v \partial u} + 2\frac{\partial f}{\partial v} + 4x^2 \frac{\partial^2 f}{\partial v^2} + 2x \frac{\partial^2 f}{\partial u \partial v}$$

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$$\frac{\partial g}{\partial y}(x,y) = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial y} = \frac{\partial f}{\partial u} - 2y \frac{\partial f}{\partial v}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial g}{\partial y}(x,y)\right) = \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial u} - 2y \frac{\partial f}{\partial v}\right] = \frac{\partial}{\partial y}\frac{\partial f}{\partial u} - 2\frac{\partial f}{\partial v} - 2x \frac{\partial}{\partial y}\frac{\partial f}{\partial v} =$$

$$= \frac{\partial^2 f}{\partial u^2} \cdot \frac{\partial u}{\partial y} + \frac{\partial^2 f}{\partial v \partial u} \cdot \frac{\partial v}{\partial y} - 2\frac{\partial f}{\partial v} - 2y\left(\frac{\partial^2 f}{\partial v^2} \cdot \frac{\partial v}{\partial y} + \frac{\partial^2 f}{\partial u \partial v} \cdot \frac{\partial u}{\partial y}\right) =$$

$$= \frac{\partial^2 f}{\partial u^2} - 2y \frac{\partial^2 f}{\partial v \partial u} - 2\frac{\partial f}{\partial v} + 4y^2 \frac{\partial^2 f}{\partial v^2} - 2y \frac{\partial^2 f}{\partial u \partial v}$$