• $U \subset \mathbb{R}^n$ aberto
$g: U \to \mathbb{R}^n$

TFC:
- g injectiva
- g classe C_1
- $\det Dg(a) \neq 0, \forall a \in U$

Localmente em $a \in U$, g é bem aproximada pela TFL:

$$Dg(a): \mathbb{R}^n \to \mathbb{R}^n$$

TFC: Transformação de coordenadas
TFL: Transformação linear

• Se $T: \mathbb{R}^n \to \mathbb{R}^n$ TFL T altera os vols-n para um factor de $|\det T|$

Logo:

$$\int_{S \subset \mathbb{R}^n} f(x)\,dx \underset{x=g(y)}{=} \int_{g^{-1}(S)} f(g(y)) \cdot |\det Dg(y)| \cdot dy$$

Teorema:

$U \subset \mathbb{R}^n$ aberto, $g: U \to \mathbb{R}^n$ TFC, $S \subset U$, $f: S \to \mathbb{R}$ tal que

Se $\int_S f$ existe então $\int_{g^{-1}(S)} f \circ g \, |\det Dg|$ também existe

$$\int_S f(x_1,\dots,x_n)\, dx_1 \dots dx_n \underset{x=g(y)}{=} \int_{g^{-1}(S)} f\circ g(y_1,\dots,y_n) \cdot |\det Dg(y)| \cdot dy_1 \dots dy_n$$

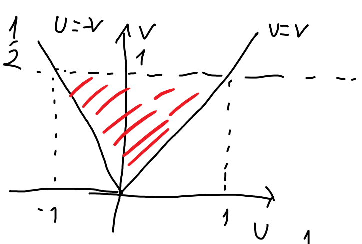
ex)

$$\int_S \exp\left(\frac{x-y}{x+y}\right) = \int_0^1 \int_0^{1-x} \exp\left(\frac{x-y}{x+y}\right) dy\, dx =$$

$$\begin{cases} u = x-y \\ v = x+y \end{cases} \to \begin{cases} x = \frac{1}{2}(u+v) \\ y = \frac{1}{2}(v-u) \end{cases} \quad (x,y) = g(u,v) = \left(\frac{1}{2}(v+u), \frac{1}{2}(v-u)\right)$$

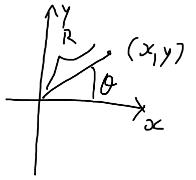
$$Dg(u,v) = \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix} \quad \det Dg(u,v) = \frac{1}{2}$$

$$S: \begin{cases} x \geq 0 \\ y \geq 0 \\ x+y \leq 1 \end{cases} \Rightarrow g^{-1}(S): \begin{cases} u+v \geq 0 \\ v-u \geq 0 \\ v \leq 1 \end{cases}$$



$$\int_0^1 \int_{-v}^{v} \exp\left(\frac{u}{v}\right) \cdot \frac{1}{2}\, du\, dv = \frac{1}{2} \int_0^1 \left[v \exp\left(\frac{u}{v}\right) \right]_{u=-v}^{u=v} dv = \frac{1}{2} \int_0^1 v(e - e^{-1})\, dv = \frac{1}{4}(e - e^{-1})$$

Coordenadas polares:



$$\begin{cases} x = R\cos(\theta) \\ y = R\operatorname{sen}(\theta) \end{cases} \qquad (x,y) = g(R,\theta) = (R\cos(\theta), R\operatorname{sen}(\theta))$$

$R = \sqrt{x^2+y^2}$

$\theta = \operatorname{arctan} \frac{y}{x}$ $(1^{\circ}Q)$

$$Dg(R,\theta) = \begin{bmatrix} \cos(\theta) & -R\operatorname{sen}(\theta) \\ \operatorname{sen}(\theta) & R\cos(\theta) \end{bmatrix}$$

$R \in]0,+\infty[\quad \theta \in]0,2\pi[$ $\qquad \det Dg = R$

$$\iint_S f(x,y)\,dx\,dy = \iint_{g^{-1}(S)} f(R\cos(\theta), R\operatorname{sen}(\theta)) \cdot R \; dR\,d\theta$$

Coordenadas cilíndricas em $\mathbb{R}^3$

$$\begin{cases} x = \rho\cos(\theta) \\ y = \rho\operatorname{sen}(\theta) \\ z = z \end{cases} \qquad \rho = \sqrt{x^2+y^2}$$

$$(x,y,z) = g(\rho,\theta,z) = (\rho\cos(\theta), \rho\operatorname{sen}(\theta), z)$$

$$Dg(\rho,\theta,z) = \begin{vmatrix} \cos(\theta) & -\rho\operatorname{sen}(\theta) & 0 \\ \operatorname{sen}(\theta) & \rho\cos(\theta) & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

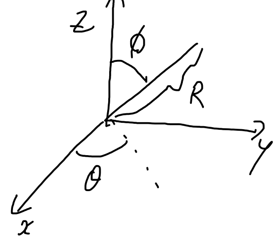
$\rho \in]0,+\infty[$ $\quad z \in \mathbb{R}$

$\theta \in]0,2\pi[$

$\det Dg = \rho$

$$\iiint_S f(x,y,z)\,dx\,dy\,dz = \iiint_{g^{-1}(S)} f(\rho\cos(\theta), \rho\operatorname{sen}(\theta), z) \cdot \rho \; d\rho\,d\theta\,dz$$

Coordenadas esféricas



$R = \sqrt{x^2+y^2+z^2}$

$$\begin{cases} x = R\cos(\theta)\operatorname{sen}(\phi) \\ y = R\operatorname{sen}(\theta)\operatorname{sen}(\phi) \\ z = R\cos(\phi) \end{cases}$$

$R \in]0;+\infty[$

$\theta \in]0;2\pi[$

$\phi \in]0;\pi[$

$$(x,y,z) = g(R,\theta,\phi) = (R\cos(\theta)\operatorname{sen}(\phi), R\operatorname{sen}\theta\operatorname{sen}\phi, R\cos\phi)$$

$$Dg = \begin{bmatrix} \cos\theta\operatorname{sen}\phi & -R\operatorname{sen}\theta\operatorname{sen}\phi & R\cos\theta\cos\phi \\ \operatorname{sen}\theta\operatorname{sen}\phi & R\cos\theta\operatorname{sen}\phi & R\operatorname{sen}\theta\cos\phi \\ \cos\phi & 0 & -R\operatorname{sen}\phi \end{bmatrix}$$

$\det Dg(R,\theta,\phi) = R^2\operatorname{sen}\phi$

$$\iiint_S f(x,y,z)\,dx\,dy\,dz =$$

$$\iiint_{g^{-1}(S)} f(R\cos\theta\operatorname{sen}\phi, R\operatorname{sen}\theta\operatorname{sen}\phi, R\cos\phi) \cdot R^2\operatorname{sen}\phi \; dR\,d\theta\,d\phi$$