1. Para cada um dos casos seguintes calcule o trabalho realizado pelo campo vectorial ao longo do caminho indicado:

   a) Campo $f : \mathbb{R}^2 \to \mathbb{R}^2$ definido por $f(x,y) = (-y, x)$ e caminho dado por $g(t) = (t\cos t, t \operatorname{sen} t)$ com $t \in [0, 2\pi]$.

   b) Campo $f : \mathbb{R}^3 \to \mathbb{R}^3$ definido por $f(x,y,z) = (x, z, z-y)$ e caminho definido por $g(t) = (t^2, \cos t, \operatorname{sen} t)$ com $t \in [0, 2\pi]$.

1) a)

$$\int f\,dg = \int_0^{2\pi} f(g(t)) \cdot g'(t)\,dt = \int_0^{2\pi} t^2\,dt = \left[\frac{t^3}{3}\right]_0^{2\pi} = \frac{(2\pi)^3}{3}$$

$$f(g(t)) = (-t \operatorname{sen}(t), t\cos(t))$$
$$g'(t) = (\cos(t) - t\operatorname{sen}(t), \operatorname{sen}(t) + t\cos(t))$$

$$> -t \cancel{\operatorname{sen}(t)\cos(t)} + t^2 \operatorname{sen}^2(t) + t\cancel{\cos(t)\operatorname{sen}(t)} + t^2\cos^2(t) = $$
$$= t^2$$

b)

$$\int f\,dg = \int_0^{2\pi} f(g(t)) \cdot g'(t)\,dt = \int_0^{2\pi} 2t^3 - 1 + \cos(t)\operatorname{sen}(t)\,dt = \left[2\cdot\frac{t^4}{4} - t + \frac{\operatorname{sen}^2(t)}{2}\right]_0^{2\pi} = 8\pi^4 - 2\pi$$

$$f(g(t)) = (t^2, \operatorname{sen}(t), \operatorname{sen}(t) - \cos(t))$$
$$g'(t) = (2t, -\operatorname{sen}(t), \cos(t))$$

$$> 2t^3 - \operatorname{sen}^2(t) + \cos(t)(\operatorname{sen}(t) - \cos(t)) =$$
$$= 2t^3 - 1 + \cos(t)\operatorname{sen}(t)$$

2. Calcule o trabalho realizado pelo campo vectorial $f(x,y,z) = (x, z, 2y)$ ao longo das seguintes curvas:

   a) O segmento de recta que une o ponto $(0,0,0)$ a $(1,2,3)$.

   b) A intersecção das superfícies $x^2+y^2 = 1$ e $z = x^2 - y^2$ num sentido que parece o anti-horário quando visto desde o ponto $(0,0,100)$.

   c) A intersecção das superfícies definidas pelas equações $x = y^2 + z^2$ e $2y + x = 3$ num sentido que parece o horário quando visto desde o ponto $(100,-1,0)$.

2.
a)

$$\int f\,dg = \int_0^1 f(g(t)) \cdot g'(t)\,dt = \int_0^1 t + 6t + 12t\,dt =$$

$g(t) = (t, 2t, 3t)$

$t \in [0,1]$

$$= \left[\frac{t^2}{2} + 3t^2 + 6t^2\right]_0^1 = \frac{1}{2} + 3 + 6 = \frac{19}{2}$$

$f(g(t)) = (t, 3t, 4t)$

$g'(t) = (1,2,3)$

b)

$$\begin{cases} x^2+y^2 = 1 \\ z = x^2 - y^2 \end{cases} \Leftrightarrow \quad r^2 = x^2+y^2 = 1 \qquad x = \cos(t) \quad y = \operatorname{sen}(t)$$

$$z = \cos^2(t) - \operatorname{sen}^2(t)$$

$$g(t) = (\cos(t), \operatorname{sen}(t), \cos^2(t) - \operatorname{sen}^2(t))$$

$$\int_0^{2\pi} -\cos(t)\operatorname{sen}(t) + \cos^3(t) - 9\cos(t)\operatorname{sen}(t)^2\,dt =$$

$$= \left[-\frac{10}{3}\operatorname{sen}^3(t) - \frac{\operatorname{sen}^2(t)}{2} + \operatorname{sen}(t)\right]_0^{2\pi} = 0$$

$$f(g(t)) = (\cos(t), \cos^2(t) - \operatorname{sen}^2(t), 2\operatorname{sen}(t))$$
$$g'(t) = (-\operatorname{sen}(t), \cos(t), -4\cos(t)\operatorname{sen}(t))$$

$$> -\cos \operatorname{sen} + \cos(\cos^2 - \operatorname{sen}^2) - 8\cos\operatorname{sen}^2 =$$
$$= -\cos\operatorname{sen} + \cos^3 - 9\cos\operatorname{sen}^2$$

2, c)

$$\begin{cases} x = y^2 + z^2 \\ 2y + x = 3 \end{cases} \Leftrightarrow \begin{cases} 3 = y^2 - 2y + 1 - 1 + z^2 \\ x = 3 - 2y \end{cases} \Leftrightarrow \begin{cases} 4 = (y-1)^2 + z^2 \\ r = \sqrt{4} = 2 \end{cases}$$

$$\begin{cases} x = 3 - 2(2\cos(\theta) - 1) = 5 - 4\cos(\theta) \\ y = 2\cos(\theta) - 1 \\ z = 2\operatorname{sen}(\theta) \end{cases}$$

$$g(t) = (5 - 4\cos(t), 2\cos(t) - 1, 2\operatorname{sen}(t))$$
$$g'(t) = (4\operatorname{sen}(t), -2\operatorname{sen}(t), 2\cos(t))$$
$$f(g(t)) = (5 - 4\cos(t), 2\operatorname{sen}(t), 4\cos(t))$$

$$> 4\operatorname{sen}(5 - 4\cos) - 4\operatorname{sen}^2 + 8\cos^2 =$$
$$= 20\operatorname{sen} - 16\operatorname{sen}\cos - 4\operatorname{sen}^2 + 8\cos^2$$

$$\int f\,dg = \int f(g(t)) \cdot g'(t)\,dt = \int_{2\pi}^{0} 20\operatorname{sen} - 16\operatorname{sen}\cos - 4\operatorname{sen}^2 + 8\cos^2\,dt =$$

$$= \left[-20\cos(t) - \frac{16}{2}\operatorname{sen}^2(t) - 2x + \operatorname{sen}(2x) + 2\operatorname{sen}(2x) + 4x\right]_{2\pi}^{0} =$$

$$= -20 + 4\pi - 8\pi + 20 = -4\pi$$

3. Para cada um dos casos seguintes determine se o campo vectorial é ou não conservativo. Em caso afirmativo, calcule um potencial.

   a) $a(x,y) = (y^2, x^3)$.

   b) $b(x,y) = (x^3 + y, y^2 + x)$.

   c) $c(x,y) = (e^x, e^y)$.

   d) $d(x,y) = \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right)$.

   e) $e(x,y,z) = (y, x, 2z)$.

   f) $g(x,y,z) = (-y, x, z)$.

a) $\frac{\partial a_1}{\partial y} = 2y \neq \frac{\partial a_2}{\partial x} = 3x^2$ Não é conserv.

b) $\frac{\partial b_1}{\partial y} = 1 = \frac{\partial b_2}{\partial x} = 1$ é fechado

$$\frac{\partial \phi}{\partial x} = x^3 + y \Leftrightarrow \phi = \frac{x^4}{4} + yx + \alpha(y)$$

$$\frac{\partial \phi}{\partial y} = x + \alpha'(y) = y^2 + x \Leftrightarrow y^2 = \alpha'(y) \Leftrightarrow \alpha(y) = \frac{y^3}{3}$$

Logo $\phi(x,y) = \frac{x^4}{4} + yx + \frac{y^3}{3}$

c) $\frac{\partial c_1}{\partial y} = \frac{\partial c_2}{\partial x} \Rightarrow$ é fechada

$$\frac{\partial \phi}{\partial x} = e^x \Leftrightarrow \phi = e^x + \alpha(y)$$

$$\frac{\partial \phi}{\partial y} = \alpha'(y) = e^y \Leftrightarrow e^y = \alpha(y)$$

Logo

$\phi(x,y) = e^x + e^y$

d) $\frac{\partial d_1}{\partial y} = \frac{-2xy}{(x^2+y^2)^2} = \frac{\partial d_2}{\partial x} = \frac{-2xy}{(x^2+y^2)^2} \Rightarrow$ é fechado

$$\frac{\partial \phi}{\partial x} = \frac{x}{x^2+y^2} \Leftrightarrow \phi = \int \frac{x}{x^2+y^2}\,dx = \frac{1}{2}\ln|x^2+y^2| + \alpha(y)$$

$$\frac{\partial \phi}{\partial y} = \frac{y}{x^2+y^2} + \alpha'(y) = \frac{y}{x^2+y^2} \Leftrightarrow \alpha'(y) = 0$$

Logo $\phi(x,y) = \frac{1}{2}\ln|x^2+y^2|$

e) $\frac{\partial e_1}{\partial y} = \frac{\partial e_2}{\partial x} = 1$

$\frac{\partial e_1}{\partial z} = \frac{\partial e_3}{\partial z} = 0 \Rightarrow$ é fechado

$\frac{\partial e_2}{\partial z} = \frac{\partial e_3}{\partial y} = 0$

$$\frac{\partial \phi}{\partial x} = y \Rightarrow \phi = yx + \alpha(y) + \beta(z)$$

$$\frac{\partial \phi}{\partial y} = x = x + \alpha'(y) + 0 \Leftrightarrow \alpha'(y) = 0$$

$$\frac{\partial \phi}{\partial z} = 2z = \beta'(y) \Leftrightarrow \beta'(y) = z^2$$

Logo $\phi(x,y,z) = xy + z^2$

f) $\frac{\partial f_1}{\partial y} = \frac{\partial f_2}{\partial x} \Leftrightarrow -1 = 1$ Não é conservativo

$\frac{\partial f_1}{\partial z} = \frac{\partial f_3}{\partial x} \Leftrightarrow 0 = 0$

$\frac{\partial f_2}{\partial z} = \frac{\partial f_3}{\partial y} \Leftrightarrow 0 = 0$

4. Considere o campo vectorial

$$F(x,y,z)=\left(\frac{x}{1+x^2+y^2},\frac{y}{1+x^2+y^2},2z\right).$$

a) Calcule o trabalho realizado pelo campo F ao longo da linha definida por

$$\{(\cos t,\operatorname{sen} t,t),\ 0\le t\le 2\pi\}.$$

b) Calcule o trabalho realizado pelo campo F ao longo da linha definida pelas equações

$$y^2+z^2=1\ ;\ x=y^2-z^2$$

segundo um sentido à sua escolha.

4 a) $\int F\,dg = \int_0^{2\pi} F(g(t))\cdot g'(t)\,dt = \int_0^{2\pi} 2t\,dt = 4\pi^2$

$F(g(t)) = \left(\frac{\cos(t)}{2}, \frac{\operatorname{sen}(t)}{2}, 2t\right)$

$g'(t) = (-\operatorname{sen}(t), \cos(t), 1)$

$> -\frac{\cos(t)\operatorname{sen}(t)}{2} + \frac{\cos(t)\operatorname{sen}(t)}{2} + 2t = 2t$

b) $\frac{\partial\phi}{\partial x} = F_1 = \frac{x}{1+x^2+y^2} \Rightarrow \phi = \frac{1}{2}\ln|1+x^2+y^2| + \alpha(y) + \beta(z)$

$\frac{\partial\phi}{\partial y} = \frac{y}{1+x^2+y^2} + \alpha'(y) = \frac{y}{1+x^2+y^2} \Rightarrow \alpha'(y)=0$

$\frac{\partial\phi}{\partial z} = 0 + \beta'(z) = 2z \Rightarrow \beta(z) = z^2$

Logo $\phi(x,y,z) = \frac{1}{2}\ln|1+x^2+y^2| + z^2$

$\int_0^{2\pi} F\,dg = \int_0^{2\pi} \nabla\phi\,dg = \phi(g(2\pi)) - \phi(g(0)) =$

$= \phi(1,1,0) - \phi(1,1,0) =$

$= 0$

$g(t) = (\cos^2(t) - \operatorname{sen}^2(t), \cos(t), \operatorname{sen}(t))$