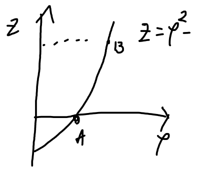
1. Sendo $F(x,y,z) = (y, -x, \cos(x^2+z^2))$, calcule o fluxo de $\nabla \times F$ através da superfície

$$S = \{(x,y,z) \in \mathbb{R}^3 : 0 < z = x^2 + y^2 - 1 < 3\}$$

no sentido da normal com terceira componente negativa.

$$\int_S \mathrm{Rot}\,F \cdot n = \int_{\partial S} F = \oint_A F + \oint_B F = -2\pi + 8\pi = 6\pi$$



$$\oint_A F = \int_0^{2\pi} F(g_A(\theta)) \cdot g_A'(\theta)\, d\theta = \int_0^{2\pi} -\sin^2(\theta) - \cos^2(\theta)\, d\theta = -2\pi$$

$$g_A = (\cos(\theta), \sin(\theta), 0)$$

$$\oint_B F = -\int_0^{2\pi} F(g_B(\theta)) \cdot g_B'(\theta)\, d\theta = -\int_0^{2\pi} 4(-\sin^2(\theta) - \cos^2(\theta))\, d\theta = 8\pi$$

$$g_B = (2\cos(\theta), 2\sin(\theta), 3)$$

2. Usando o teorema de Stokes, calcule o trabalho realizado pelo campo $G(x,y,z) = (x, -z, y+z^2)$, ao longo da linha definida pelas equações $x^2+z^2 = 1$; $y+z = 1$ e orientada no sentido horário quando vista do ponto $(0, 100, 0)$.

$$\int_{\partial S} G = \int_S \mathrm{Rot}(G) \cdot n \qquad \mathrm{Rot}(G) = (1-(-1), 0-0, 0-0) = (2,0,0)$$

$$g(\theta, y) = (\cos(\theta), y, -\sin(\theta))$$

$$D_g = \begin{bmatrix} -\sin(\theta) & 0 \\ 0 & 1 \\ -\cos(\theta) & 0 \end{bmatrix} \qquad \frac{\partial g}{\partial \theta} \times \frac{\partial g}{\partial y} = \begin{vmatrix} e_1 & e_2 & e_3 \\ -\sin(\theta) & 0 & -\cos(\theta) \\ 0 & 1 & 0 \end{vmatrix} = (\cos(\theta), 0, -\sin(\theta))$$

$$\int_S (2,0,0) \cdot (\cos(\theta), 0, -\sin(\theta)) = \int_y \int_0^{2\pi} 2\cos(\theta)\, d\theta = 0$$

3. Usando o teorema de Stokes, calcule o trabalho realizado pelo campo vectorial

$$H(x,y,z) = (x^2 - y, y^2 - x, y^2 - x^2 + z^3)$$

ao longo do caminho

$$g(t) = (\cos t, \operatorname{sen} t, \cos 2t) ; \quad t \in [0, 2\pi].$$

$$\int_g H = \int_S \mathrm{Rot}(H) \cdot n$$

$$\mathrm{Rot}(H) = (2y - 0, 2x, -1-(-1)) = (2y, 2x, 0)$$

$$\int_S \mathrm{Rot}(H) \cdot n = \int_S (2\sin(\theta), 2\cos(\theta), 0) \cdot (\cos(\theta), \sin(\theta), 0) = \int_S 4\cos(\theta)\sin(\theta) =$$

$$\int_z \int_0^{2\pi} 4\cos(\theta)\sin(\theta)\, d\theta = 0$$

$$g(\theta, z) = (\cos(\theta), \sin(\theta), z)$$

$$\frac{\partial g}{\partial \theta} \times \frac{\partial g}{\partial z} = \begin{vmatrix} e_1 & e_2 & e_3 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{vmatrix} = (\cos(\theta), \sin(\theta), 0)$$

4. Considere a superfície

$$S = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1 ; z > 0\},$$

orientada com a normal unitária n tal que $n_z > 0$. Seja $G(x,y,z) = (xz, yz, 1-z^2)$. Calcule o fluxo $\int_S G \cdot n$:

a) Pelo teorema da divergência.

b) Pelo teorema de Stokes.

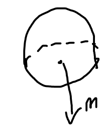
a) $\int_S \mathrm{div}\,G = \int_{\partial S} G \cdot n_{ext} = \int_S G \cdot n + \int_A G \cdot n$

$$\Downarrow$$

$$\int_S G \cdot n = \int \mathrm{div}\,G - \int_A G \cdot n = 0 - (-\pi) = \pi$$

$$\int_S \mathrm{div}\,G = 0 \quad \mathrm{div}\,G = z + z - 2z = 0$$

$\boxed{A}$



$$\int_A G \cdot (0,0,-1) = -\int_A 1 - z^2 = \int_A -1 = -\text{Area de } A = -\pi$$

b) $\int_S G \cdot n = \int_S \mathrm{Rot}(f) \cdot n = \oint_{\partial S} f = \pi$

Seja $G = \mathrm{Rot}(f)$

$$\begin{cases} \partial_2 f_3 - \partial_3 f_2 = G_1 = xz \\ \partial_3 f_1 - \partial_1 f_3 = G_2 = yz \\ \partial_1 f_2 - \partial_2 f_1 - G_3 = 1 - z^2 \end{cases}$$

$$\frac{\partial f_2}{\partial z} = -xz \Rightarrow f_2 = -\frac{xz^2}{2} + \alpha(x,y)$$

$$\frac{\partial f_1}{\partial z} = yz \Rightarrow f_1 = \frac{yz^2}{2} + \beta(x,y)$$

$$\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} = 1 - z^2 \Leftrightarrow -\frac{z^2}{2} - \frac{z^2}{2} + \alpha'(x,y) + \beta'(x,y) = 1 - z^2 \Leftrightarrow$$

$$\Leftrightarrow \alpha'(x,y) + \beta'(x,y) = 1$$

$$\alpha = 0 \qquad f = \left(-\frac{xz^2}{2}, \frac{yz^2}{2} + x, 0\right)$$
$$\beta = x$$

$$\oint_{\partial S} f = \int_0^{2\pi} f(g(\theta)) \cdot g'(\theta)\, d\theta = \int_0^{2\pi} (0, \cos(\theta), 0) \cdot (-\sin(\theta), \cos(\theta), 0)\, d\theta =$$

$$\partial S : g(\theta) = (\cos(\theta), \sin(\theta), 0)$$

$$\int_0^{2\pi} \cos(\theta)^2\, d\theta = \pi$$

5. Considere a superfície

$$S = \{(x,y,z) \in \mathbb{R}^3 : z^2 + (\sqrt{x^2+y^2} - 2)^2 = 1 \; ; \; x > 0\},$$

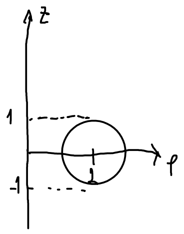
orientada com a normal unitária n à sua escolha. Seja $F(x,y,z) = (1, 2z, 2xy)$. Calcule o fluxo $\int_S F \cdot n$:

a) Pelo teorema da divergência.

b) Pelo teorema de Stokes.

a) $\int_S \operatorname{div} F = \int_{\partial S} F \cdot n =$

$= \int_S F\cdot n + \int_A F \cdot n + \int_B F \cdot n$



$\rho^2 = x^2 + y^2$

$z^2 + (\rho - 2)^2 = 1$

$\int_S F\cdot n = \int_G \operatorname{div} F - \int_A F\cdot n - \int_B F \cdot n = -2\pi$

$\int_S \operatorname{div} F = 0$

$\operatorname{div} F = 0 + 0 + 0 = 0$

$\boxed{A}$ $\vec{n}(1,0,0)$

$\int_A (1, 2z, 2xy)\cdot(1,0,0) = \int_A 1 = \text{Área de } A = \pi$

$\boxed{B}$ $\vec{n}(1,0,0)$

$\int_B (1, 2z, 2xy)\cdot(1,0,0) = \int_B 1 = \text{Área de } B = \pi$

b) $\int_S F\cdot n = \int_S \operatorname{Rot}(f)\cdot n = \oint_{\partial S} f$

$F = \operatorname{Rot}(f)$

$$\begin{cases} \partial_2 f_3 - \partial_3 f_2 = 1 \\ \partial_3 f_1 - \partial_1 f_3 = 2z \\ \partial_1 f_2 - \partial_2 f_1 = 2xy \end{cases}$$

$-\frac{\partial f_2}{\partial z} = 1 \rightsquigarrow f_2 = -z + \alpha(x,y)$

$\frac{\partial f_1}{\partial z} = 2z \rightsquigarrow f_1 = z^2 + \beta(x,y)$

$\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} = xy \Leftrightarrow \frac{\partial \alpha}{\partial x} - \frac{\partial \beta}{\partial y} = 2xy \Leftrightarrow$

$\beta = 0$

$\alpha = \frac{x^2 y}{2}$

$f(z^2, -z + x^2 y, 0)$

$g(\theta)\,(0, \cos(\theta), \sin(\theta))$

$\oint_{\partial S} f = \oint_A f + \oint_B f = 2\int_0^{2\pi} f(g)\cdot g'(\theta)\, d\theta = 2\int_0^{2\pi} (\sin^2(\theta), -\sin(\theta), 0)\cdot(0, -\sin(\theta), \cos(\theta))\, d\theta =$

$= 2\int_0^{2\pi} \sin^2(\theta)\, d\theta = 2\pi$

6. Considere o campo vectorial

$$H(x,y,z) = \left(\frac{z}{x^2+z^2} + x, y, \frac{-x}{x^2+z^2} + z\right).$$

a) Calcule o trabalho de H ao longo da elipse definida por $2(x-1)^2 + \frac{y^2}{4} = 1$, $z = 0$, percorrida no sentido horário para um observador colocado no ponto $(1, 0, 100)$.

b) Calcule o trabalho de H ao longo da linha definida por $x^2 + z^2 = 2$, $y + z = 1$, percorrida num sentido à sua escolha.

c) Será H um gradiente no seu domínio?

a) $g(\theta, \varphi) = \left(\frac{\cos(\theta)}{\sqrt{2}} + 1, 2\sin(\theta), 0\right)$

$z = 0 \quad \varphi = 1$

$y = 2\sin(\theta)$

$x = \frac{\cos(\theta)}{\sqrt{2}} + 1$

$\oint_g H = -\int_0^{2\pi} H(g)\cdot g'\, d\theta = -\int_0^{2\pi} \left(\frac{\cos(\theta)}{\sqrt{2}} + 1, 2\sin(\theta), -\frac{1}{\frac{\cos(\theta)}{\sqrt{2}} + 1}\right)\cdot\left(-\frac{1}{\sqrt{2}}\sin(\theta), 2\cos(\theta), 0\right) d\theta =$

$= \int_0^{2\pi} -\frac{\cos \sin}{2} - \frac{\sin}{\sqrt{2}} + 4\cos\sin\, d\theta = 0$

b) $x^2 + z^2 = \rho^2 = 2$

$\rho = \sqrt{2}$

$y = 1 - z$

$x = \sqrt{2}\cos(\theta)$

$z = \sqrt{2}\sin(\theta)$

$y = 1 - \sqrt{2}\sin(\theta)$

$g(\theta) : (\sqrt{2}\cos(\theta), 1 - \sqrt{2}\sin(\theta), \sqrt{2}\sin(\theta))$

$\oint_g H = \int H(g)\cdot g'\, d\theta = \int_0^{2\pi} \left(\frac{\sqrt{2}}{2}\sin(\theta) + \sqrt{2}\cos(\theta), 1 - \sqrt{2}\sin(\theta), -\frac{\sqrt{2}}{2}\cos(\theta) + \sqrt{2}\sin(\theta)\right)\cdot$

$(-\sqrt{2}\sin(\theta), -\sqrt{2}\cos(\theta), \sqrt{2}\cos(\theta))$

$= \int_0^{2\pi} -\sin^2(\theta) - \cancel{2\sin(\theta)\cos(\theta)} \underbrace{- \sqrt{2}\cos(\theta)}_{\to 0} + \cancel{2\sin(\theta)\cos(\theta)} - \cos^2(\theta) + \underbrace{2\sin(\theta)\cos(\theta)}_{\to 0}\, d\theta =$

$= \int_0^{2\pi} -1\, d\theta = -2\pi$

c) $\phi = \operatorname{arctan}\left(\frac{x}{z}\right) + \frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2}$ logo não é gradiente

tal que $\nabla\phi = H$