1. Calcule o gradiente e a matriz Hessiana de cada uma das funções seguintes:

a) $f(x,y) = x \arctan y$

b) $f(x,y,z) = \ln x + \ln y + e^z$

1 a) $f(x,y) = x \arctan(y)$

$$\nabla f(x,y) = \left[\arctan(y) \quad \frac{x}{1+y^2}\right]$$

$$\frac{\partial \nabla f(x,y)}{\partial x} = \left[0 \quad \frac{1}{1+y^2}\right]$$

$$\frac{\partial \nabla f(x,y)}{\partial y} = \left[\frac{1}{1+y^2} \quad -\frac{2y}{(y^2+1)^2}\right]$$

$$H_f = \begin{bmatrix} 0 & \frac{1}{1+y^2} \\ \frac{1}{1+y^2} & \frac{-2yx}{(y^2+1)^2} \end{bmatrix}$$

b) $f(x,y,z) = \ln(x) + \ln(y) + e^z$

$$\nabla f(x,y,z) = \left[\frac{1}{x} \quad \frac{1}{y} \quad e^z\right]$$

$$\frac{\partial \nabla f(x,y,z)}{\partial x} = \left[-\frac{1}{x^2} \quad 0 \quad 0\right]$$

$$\frac{\partial \nabla f(x,y,z)}{\partial y} = \left[0 \quad -\frac{1}{y^2} \quad 0\right]$$

$$\frac{\partial \nabla f(x,y,z)}{\partial z} = \left[0 \quad 0 \quad e^z\right]$$

$$H_f = \begin{bmatrix} -\frac{1}{x^2} & 0 & 0 \\ 0 & -\frac{1}{y^2} & 0 \\ 0 & 0 & e^z \end{bmatrix}$$

2. Mostre que a função $V(x,y,z) = \dfrac{1}{\sqrt{x^2+y^2+z^2}}$ verifica a equação de Laplace:

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 ; \quad (x,y,z) \neq (0,0,0)$$

$$\frac{\partial V}{\partial x} = \left(-\frac{1}{2}\right)(x^2+y^2+z^2)^{-\frac{3}{2}}(2x) = -\frac{x}{(x^2+y^2+z^2)^{\frac{3}{2}}}$$

$$\frac{\partial^2 V}{\partial x^2} = \frac{3x^2}{(x^2+z^2+y^2)^{5/2}} - \frac{1}{(x^2+z^2+y^2)^{3/2}} = \frac{2x^2-z^2-y^2}{(x^2+z^2+y^2)^{5/2}}$$

$$\frac{\partial V}{\partial y} = -\frac{y}{(x^2+y^2+z^2)^{\frac{3}{2}}} \qquad \frac{\partial^2 V}{\partial y^2} = \frac{2y^2-z^2-x^2}{(x^2+z^2+y^2)^{5/2}}$$

$$\frac{\partial V}{\partial z} = -\frac{z}{(x^2+y^2+z^2)^{\frac{3}{2}}} \qquad \frac{\partial^2 V}{\partial z^2} = \frac{2z^2-x^2-y^2}{(x^2+z^2+y^2)^{5/2}}$$

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = \frac{\cancel{2x^2} + \cancel{2y^2} + \cancel{2z^2} - \cancel{x^2} - \cancel{x^2} - \cancel{y^2} - \cancel{y^2} - \cancel{z^2} - \cancel{z^2}}{(x^2+z^2+y^2)^{5/2}} = 0 //$$

3. Seja $w(x,y) = f(y-x, x+y)$, em que $f : \mathbb{R}^2 \to \mathbb{R}$ é uma função de classe C^2. Mostre que se tem

$$4\frac{\partial^2 f}{\partial u \partial v} = \frac{\partial^2 w}{\partial y^2} - \frac{\partial^2 w}{\partial x^2},$$

em que $u = y - x$ e $v = x + y$.

$$\frac{\partial w}{\partial x} = \frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\cdot(-1) + \frac{\partial f}{\partial v} = \frac{\partial f}{\partial v} - \frac{\partial f}{\partial u}$$

$$\frac{\partial^2 w}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial v} - \frac{\partial f}{\partial u}\right) = \frac{\partial^2 f}{\partial x \partial v} - \frac{\partial^2 f}{\partial x \partial u} = -2\frac{\partial^2 f}{\partial u \partial v}$$

$$\frac{\partial w}{\partial y} = \frac{\partial f}{\partial y} = \frac{\partial f}{\partial u}\cdot\overbrace{\boxed{\frac{\partial u}{\partial y}}}^{1} + \frac{\partial f}{\partial v}\cdot\overbrace{\boxed{\frac{\partial v}{\partial y}}}^{1} = \frac{\partial f}{\partial u} + \frac{\partial f}{\partial v}$$

$$\frac{\partial^2 w}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial u} + \frac{\partial f}{\partial v}\right) = \frac{\partial^2 f}{\partial y \partial u} + \frac{\partial^2 f}{\partial y \partial v} = 2\frac{\partial^2 f}{\partial u \partial v}$$

$$\Bigg\} \; 4\frac{\partial^2 f}{\partial u \partial v}$$

4. Determine e classifique os pontos de estacionaridade de cada uma das funções seguintes:

a) $f(x, y) = x^2 - y^2 + xy$

b) $f(x, y) = x^2 + y^2 - \frac{x^3}{3}$

a)

$$\begin{cases} 2x + y = 0 \\ x - 2y = 0 \end{cases} \Leftrightarrow \begin{cases} 2x = -y \\ x = 2y \end{cases} \Leftrightarrow \begin{cases} 4y = -y \\ - \end{cases} \Leftrightarrow \begin{cases} x = 0 \\ y = 0 \end{cases}$$

$$\nabla f(x,y) = \begin{bmatrix} 2x + y & x - 2y \end{bmatrix}$$

$$\frac{\partial \nabla f(x,y)}{\partial x} = \begin{bmatrix} 2 & 1 \end{bmatrix}$$

$$\frac{\partial \nabla f(x,y)}{\partial y} = \begin{bmatrix} 1 & -2 \end{bmatrix}$$

$$Hf = \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}$$

No ponto $(0,0)$ é um ponto de sela

$\det < 0$, logo $\lambda_1 < 0 \wedge \lambda_2 > 0$ e $\lambda_1 > 0 \wedge \lambda_2 < 0$

b) $\nabla f(x,y) = \begin{bmatrix} 2x - x^2 & 2y \end{bmatrix}$

$$\begin{cases} 2x - x^2 = 0 \\ 2y = 0 \end{cases} \Leftrightarrow \begin{cases} x = 0 \vee x = 2 \\ y = 0 \end{cases}$$

Pontos: $(0,0)$ $(2,0)$

$$\frac{\partial \nabla f(x,y)}{\partial x} = \begin{bmatrix} 2 - 2x & 0 \end{bmatrix}$$

$$\frac{\partial \nabla f(x,y)}{\partial y} = \begin{bmatrix} 0 & 2 \end{bmatrix}$$

$$Hf = \begin{bmatrix} 2 - 2x & 0 \\ 0 & 2 \end{bmatrix}$$

$(0,0) \rightarrow \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ $\lambda_1, \lambda_2 > 0$ $(0,0)$ é um mínimo

$(2,0) \rightarrow \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}$ $\lambda_1 = -2$, $\lambda_2 = 2$ $(2,0)$ Ponto de sela

c) $f(x, y) = e^{1+xy}$

c) $\nabla f(x,y) = \begin{bmatrix} y e^{1+xy} & x e^{1+xy} \end{bmatrix}$

$$\begin{cases} y e^{1+xy} = 0 \\ x e^{1+xy} = 0 \end{cases} \Leftrightarrow \begin{cases} y = 0 \\ x = 0 \end{cases} \quad (0,0)$$

$$\frac{\partial \nabla f(x,y)}{\partial x} = \begin{bmatrix} y^2 e^{1+xy} & e^{1+xy} + yx e^{1+xy} \end{bmatrix}$$

$$\frac{\partial \nabla f(x,y)}{\partial y} = \begin{bmatrix} e^{1+xy} + xy e^{1+xy} & x^2 e^{1+xy} \end{bmatrix}$$

$$Hf = \begin{bmatrix} y^2 e^{1+xy} & (e yx + e) e^{yx} \\ (e yx + e) e^{yx} & x^2 e^{1+xy} \end{bmatrix}$$

$\rightarrow (0,0)$

$\begin{vmatrix} -\lambda & e \\ e & -\lambda \end{vmatrix} = 0 \Leftrightarrow \lambda^2 - e^2 = 0 \Leftrightarrow \lambda = \pm e$

$\lambda_1 = e$ $\lambda_2 = -e$

$(0,0)$ ponto em sela

d) $f(x, y) = \frac{x^2}{2} + \frac{y^2}{2} + \frac{1}{x} + \frac{1}{y}$

d) $\nabla f(x,y) = \begin{bmatrix} x - \frac{1}{x^2} & y - \frac{1}{y^2} \end{bmatrix}$

$$\begin{cases} x - \frac{1}{x^2} = 0 \\ y - \frac{1}{y^2} = 0 \end{cases} \Leftrightarrow \begin{cases} x^3 = 1 \\ y^3 = 1 \end{cases} \rightarrow (1,1)$$

$$Hf = \begin{bmatrix} 1 + \frac{2}{x^3} & 0 \\ 0 & 1 + \frac{2}{y^3} \end{bmatrix} \xrightarrow{(1,1)} \begin{bmatrix} 1+2 & 0 \\ 0 & 1+2 \end{bmatrix} \rightarrow \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

$\lambda_1 = 3$, $\lambda_2 = 3$ $(1,1)$ Mínimo local

e) $f(x, y, z) = xz - x^2 - y^2$

f) $f(x, y) = x^3 - y^4$

e) $\nabla f(x,y,z) = \begin{bmatrix} z - 2x & -2y & x \end{bmatrix}$

$$\begin{cases} z - 2x = 0 \\ -2y = 0 \\ x = 0 \end{cases} \Leftrightarrow \begin{cases} z = 0 \\ y = 0 \\ x = 0 \end{cases}$$

$$Hf = \begin{bmatrix} -2 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$\begin{vmatrix} -2-\lambda & 0 & 1 \\ 0 & -2-\lambda & 0 \\ 1 & 0 & -\lambda \end{vmatrix} = 0 \Leftrightarrow -(-2-\lambda)^2 \lambda - (-2-\lambda) = 0 \Leftrightarrow$

$\Leftrightarrow -(-2-\lambda)\left((-2-\lambda)\lambda + 1\right) = 0 \Leftrightarrow$

$\Leftrightarrow \lambda = -2 \vee -\lambda^2 - 2\lambda + 1 = 0$

$\lambda = \frac{2 \pm \sqrt{4+4}}{-2} \Leftrightarrow \lambda = \frac{2 + \sqrt{8}}{-2} \vee \lambda = \frac{2 - \sqrt{8}}{-2}$

$\lambda = -2$

$\lambda = -1 - \sqrt{2}$

$\lambda = -1 + \sqrt{2}$

Ponto em sela

f) $\nabla f(x,y) = \begin{bmatrix} 3x^2 & -4y^3 \end{bmatrix}$ $\begin{cases} 3x^2 = 0 \\ -2y = 0 \end{cases} \Leftrightarrow \begin{cases} x = 0 \\ y = 0 \end{cases}$

$$Hf = \begin{bmatrix} 6x & 0 \\ 0 & -4y^3 \end{bmatrix} \xrightarrow{(0,0)} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\begin{vmatrix} -\lambda & 0 \\ 0 & -\lambda \end{vmatrix} = 0 \Leftrightarrow$

$\Leftrightarrow \lambda^2 = 0 \Leftrightarrow$

$\lambda = 0$

$\lambda = 0$

$(0,0)$ Pode ser sela ou máximo ou mínimo

Ao substituir $(0,0)$ em $f(x,y)$ chega-se à conclusão que é ponto de sela

g) $f(x, y) = x^3 - y^2$

h) $f(x, y) = \frac{y^2}{2} + xy + x^4$

g) $\nabla f(x,y) = \begin{bmatrix} 3x^2 & -2y \end{bmatrix}$ $\begin{cases} x = 0 \\ y = 0 \end{cases}$

$$Hf = \begin{bmatrix} 6x & 0 \\ 0 & -2 \end{bmatrix} \xrightarrow{(0,0)} \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix}$$

$\begin{vmatrix} -\lambda & 0 \\ 0 & -2-\lambda \end{vmatrix} = 0$

$-\lambda(-2-\lambda) = 0$

$\lambda = 0$

$\lambda = -2$

Pode ser sela ou máximo

Ao substituir $(0,0)$ em $f(x,y)$ obtemos que $(0,0)$ é ponto em sela

h) $\nabla f(x,y) = \begin{bmatrix} y + 4x^3 & y + x \end{bmatrix}$

$$\begin{cases} y + 4x^3 = 0 \\ y + x = 0 \end{cases} \Leftrightarrow \begin{cases} -x + 4x^3 = 0 \\ y = -x \end{cases} \Leftrightarrow \begin{cases} x(-1 + 4x^2) = 0 \\ - \end{cases}$$

$$\begin{cases} x = 0 \vee x = \sqrt{\frac{1}{4}} \\ y = -x \end{cases} \Leftrightarrow \begin{cases} x = 0, x = \frac{1}{2}, x = -\frac{1}{2} \\ y = 0, y = -\frac{1}{2}, y = \frac{1}{2} \end{cases}$$

$(0,0)$; $(\frac{1}{2}, -\frac{1}{2})$; $(-\frac{1}{2}, \frac{1}{2})$

$$Hf = \begin{bmatrix} 12x^2 & 1 \\ 1 & 1 \end{bmatrix}$$

Para $(0,0)$

$\begin{vmatrix} -\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = 0 \Leftrightarrow -\lambda(1-\lambda) - 1 = 0 \Leftrightarrow$

$\Leftrightarrow \lambda^2 - \lambda - 1 = 0 \Leftrightarrow$

$\Leftrightarrow \lambda_1 = \frac{-\sqrt{5}+1}{2} \wedge \lambda_2 = \frac{+\sqrt{5}+1}{2}$

$\lambda_1 < 0$ e $\lambda_2 > 0$ logo $(0,0)$ é ponto de sela

Para $(\frac{1}{2}, -\frac{1}{2})$ e $(-\frac{1}{2}, \frac{1}{2})$

$(-\frac{1}{2})^2 = (\frac{1}{2})^2 = \frac{1}{4}$

$12 \times \frac{1}{4} = 3$

$\begin{vmatrix} 3-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = 0 \Leftrightarrow$

$(3-\lambda)(1-\lambda) - 1 = 0 \Leftrightarrow$

$\Leftrightarrow \lambda^2 - 4\lambda + 2 = 0 \Leftrightarrow$

$\Leftrightarrow \lambda = \sqrt{2} - 2 \vee \lambda = \sqrt{2} + 2$

$\lambda_1, \lambda_2 > 0$, $(-\frac{1}{2}, \frac{1}{2})$; $(\frac{1}{2}, -\frac{1}{2})$ são mínimos locais