1. Escreva o integral $\int\int_S f(x,y)dxdy$ em coordenadas polares considerando as seguintes regiões S.

(a) $S=\{(x,y)\in\mathbb{R}^2: x^2+y^2\le 2, x>|y|\}$.

(b) $S=\{(x,y)\in\mathbb{R}^2: 1\le x^2+y^2\le 4, y>x\}$.

(c) $S=\{(x,y)\in\mathbb{R}^2: 0\le x\le 1, -\sqrt{1-x^2}\le y\le x\}$.

2. Utilizando coordenadas polares (possivelmente modificadas), calcule

(a) $\int_0^1\left(\int_0^{\sqrt{1-x^2}} e^{-x^2-y^2}dy\right)dx$.

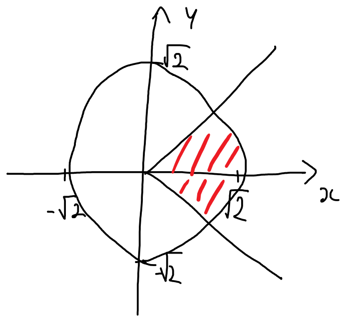
(b) $\int_0^1\left(\int_x^{\sqrt{2-x^2}} \frac{1}{1+x^2+y^2}dy\right)dx$.

(c) $\int\int_U (x^2+y^2-1)dxdy$, sendo $U=\{(x,y)\in\mathbb{R}^2: (x+1)^2+y^2\le 1 ; y>0\}$.

(d) $\int\int_S \text{sen}((x-1)^2+y^2)dxdy$, sendo $S=\{(x,y)\in\mathbb{R}^2: (x-1)^2+y^2\le \frac{\pi^2}{4}\}$.

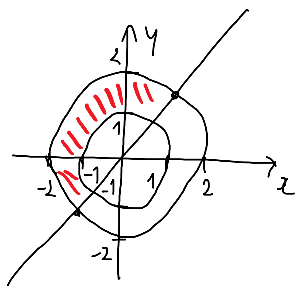
(e) A área da região $A=\{(x,y)\in\mathbb{R}^2: \frac{x^2}{4}+y^2<1 ; x>|y|\}$. ?

a)



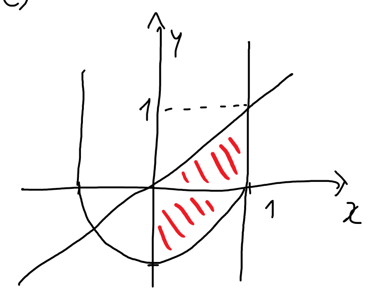
$$\int_0^{\sqrt{2}}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} f(r\cos\theta, r\sin\theta)\, r \, d\theta\, dr$$

b)



$$\int_1^2\int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} f(r\cos\theta, r\sin\theta)\, r\, d\theta\, dr$$

c)



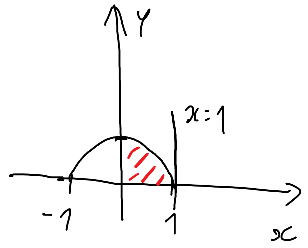
$$\int_0^1\int_{-\frac{\pi}{2}}^{0} f(r\cos(\theta), r\sin(\theta))\, r\, d\theta\, dr + \int_0^{\frac{\pi}{2}}\int_0^{\frac{1}{\cos(\theta)}} f(r\cos(\theta), r\sin(\theta))\, r\, dr\, d\theta$$

2. a)

$$\begin{cases} x = R\cos(\theta) \\ y = R\sin(\theta)\end{cases}$$

$R=\sqrt{x^2+y^2}$

$e^{-x^2-y^2}=e^{-R^2}$

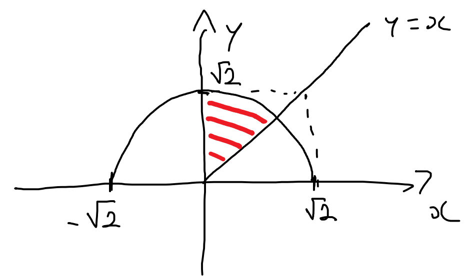


$$\int_0^{\frac{\pi}{2}}\int_0^1 e^{-R^2} R\, dR\, d\theta = \int_0^{\frac{\pi}{2}}\left[-\frac{e^{-R^2}}{2}\right]_0^1 d\theta = \left[\frac{\theta}{2}\left(1-e^{-1}\right)\right]_0^{\frac{\pi}{2}} = \frac{\pi}{4}(1-e^{-1})$$

$y=\sqrt{2-x^2} \Leftrightarrow y^2+x^2=2$

b)

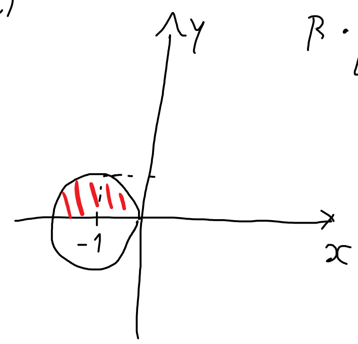
$$\begin{cases} x = R\cos(\theta) \\ y = R\sin(\theta)\end{cases}$$



$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^{\sqrt{2}} \frac{R}{1+R^2}\, dR\, d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\left[\frac{\ln(R^2+1)}{2}\right]_0^{\sqrt{2}} d\theta =$$

$$= \left[\frac{\ln(3)}{2}\theta\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} = \frac{\ln(3)}{2}\times\frac{\pi}{2} - \frac{\ln(3)}{2}\times\frac{\pi}{4} = \frac{\ln(3)\pi}{8}$$

c)



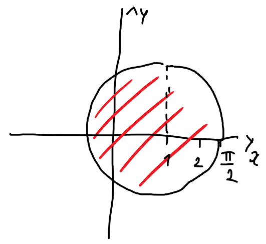
$$R\cdot\left[(R\cos(\theta))^2+(R\sin(\theta))^2-2R\cos(\theta)\right] = \left[R^2-2R\cos(\theta)\right]R = R^3-2R^2\cos(\theta)$$

$R^2=(x+1)^2+y^2$

$$\begin{cases} x+1 = R\cos(\theta) \\ y = R\sin(\theta)\end{cases}$$

$$\int_0^{\pi}\int_0^1\left[(R\cos(\theta)-1)^2+(R\sin(\theta))^2-1\right]R\, dR\, d\theta = \int_0^{\pi} \frac{1}{4} - 2\frac{1}{3}\cos(\theta)\, d\theta = \frac{\pi}{4} - \frac{2\sin(\pi)}{3} + \frac{2\sin(0)}{3} = \frac{\pi}{4}$$

2 d)



$(x-1) = R\cos(\theta)$ a soma de 1 não afeta o jacobiano

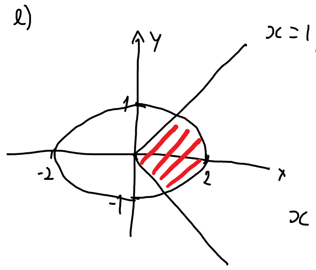
$y = R\cos(\theta)$

$R^2 = (x-1)^2 + y^2$

$$\int_0^{2\pi}\int_0^{\frac{\pi}{2}} \sin(R^2) R \, dR \, d\theta = \int_0^{2\pi}\left[-\frac{\cos(R^2)}{2}\right]_0^{\frac{\pi}{2}} d\theta =$$

$$\int_0^{2\pi} \frac{1-\cos(\frac{\pi^2}{4})}{2} d\theta = \pi\left(1-\cos\left(\frac{\pi^2}{4}\right)\right)$$

e)



$x = |y|$

$\frac{x^2}{4} + y^2 < 1$

$x > |y|$

$$\begin{cases} x = 2R\cos(\theta) \\ y = R\sin(\theta) \end{cases}$$

$\frac{x^2}{2^2} + y^2 \leq 1$ $\quad \theta = \arctan\left(\frac{y}{x}\frac{a}{b}\right) =$

$= \arctan\left(\frac{y}{x} 2\right)$

$$2\times\int_0^1\int_0^{\arctan(2)} 2R \, d\theta \, dR = 4\int_0^1 \arctan(2) R \, dR = 4\times\arctan(2)\times\frac{1}{2}$$

$$= 2\arctan(2)$$

3. Considere a transformação de coordenadas definida por

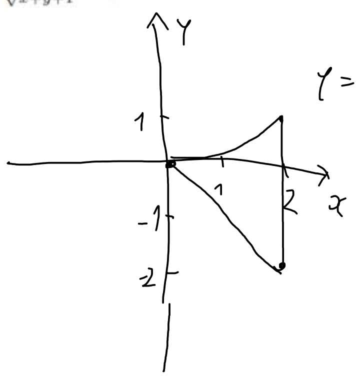
$$x = 2u + v, \qquad y = u^2 - v.$$

(a) Sendo T o triângulo com vértices $(0,0), (1,0)$ e $(0,2)$ no plano uv, determine a imagem de T no plano xy pela transformação de coordenadas.

(b) Sendo S o conjunto determinado na alínea anterior, calcule $\int\int_S \frac{1}{\sqrt{x+y+1}} dx dy$.

$$\begin{cases} x = 2u + v \\ y = u^2 - v \end{cases} \Leftrightarrow \begin{cases} y = \left(\frac{x-v}{2}\right)^2 - v \\ y = u^2 + 2u - x \end{cases}$$

$S = \{0 < x < 2, -x \leq y \leq \frac{x^2}{4}\}$



$y = \frac{x^2}{4} \Leftrightarrow \sqrt{4y} = x$

$$\int\int_S \frac{1}{\sqrt{x+y+1}} dx dy = \int_{-2}^{0}\int_{-y}^{2} \frac{1}{\sqrt{x+y+1}} dx dy + \int_0^1\int_{\sqrt{4y}}^{2} \frac{1}{\sqrt{x+y+1}} dx dy =$$

$$\int_{-2}^{0}\left[2\sqrt{x+y+1}\right]_{-y}^{2} + \int_0^1 \left[2\sqrt{x+y+1}\right]_{\sqrt{4y}}^{2} = \int_{-2}^{0} 2\sqrt{y+3} - 2 \, dy + \int_0^1 2\sqrt{y+3} - 2\sqrt{\sqrt{4y}+y+1} \, dy =$$

$$= \frac{4\cdot 3^{\frac{3}{2}} - 16}{3} - \frac{4\cdot 3^{\frac{3}{2}} - 22}{3} = 2$$

4. Considere o conjunto

$$D = \{(x,y) \in \mathbb{R}^2 : 1 < x + y < 2 \, ; \, 0 < x < y\},$$

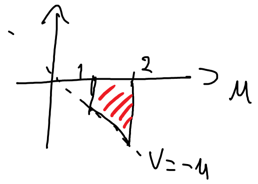
e seja $f : D \to \mathbb{R}$ definida por $f(x,y) = (y^2 - x^2)\cos(x+y)^4$. Calcule $\int_D f$ utilizando uma transformação de coordenadas apropriada. Justifique cuidadosamente.

$1 < x + y < 2$

$0 < x < y \Leftrightarrow x - y < 0$

$$\begin{cases} x + y = u \\ x - y = v \end{cases} \Leftrightarrow \begin{cases} x = \frac{u+v}{2} \\ y = \frac{u-v}{2} \end{cases}$$

$$D = \begin{cases} 1 < u < 2 \\ \frac{v-u}{2} < v < 0 \end{cases}$$



$$D_g = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

$\det D_g = -\frac{1}{2}$

$$\int_1^2\int_{-u}^{0} -uv\cos(u^4)\cdot\frac{1}{2} \, dv \, du = \int_1^2 -\frac{1}{2}u\cos(u^4)\cdot\frac{u^2}{2} du =$$

$$= \int_1^2 -\frac{u^3}{4}\cos(u^4) du = -\frac{1}{16}\left[-\sin(u^4)\right]_1^2 = -\frac{1}{16}\left(-\sin(16) + \sin(1)\right)$$

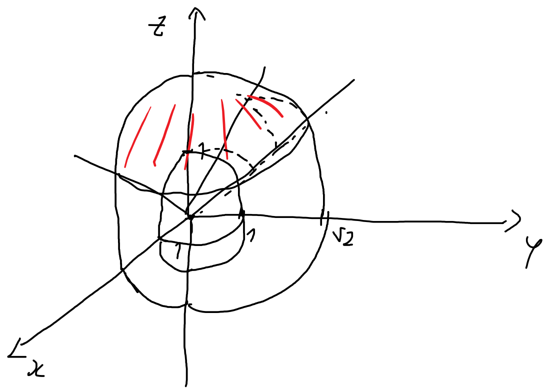
5. Use coordenadas cilíndricas ou coordenadas esféricas para exprimir o volume de cada uma das seguintes regiões em termos de um só integral iterado:

(a) $V = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 < z < \sqrt{2 - x^2 - y^2}\}$.

(b) $V = \{(x,y,z) \in \mathbb{R}^3 : y > 0, \, 1 \leq x^2 + y^2 + z^2 \leq 2, \, z > \sqrt{x^2 + y^2}\}$.

5 a)

$$\int_0^{2\pi}\int_0^1\int_{\rho^2}^{\sqrt{2-\rho^2}} \rho \, dz \, d\rho \, d\theta$$



b) $1 \leq R^2 \leq 2$

$$\int_0^{\pi}\int_1^{\sqrt{2}}\int_0^{\pi/4} R^2\sin\phi \, d\phi \, dR \, d\theta$$

6. Calcule o momento de inércia do sólido

$$U = \{(x,y,z) \in \mathbb{R}^3 : y^2+z^2 \le 1 \,;\, 0 \le x \le (y^2+z^2)^{\frac{1}{4}} \,;\, y \ge 0 \,;\, z \ge 0\},$$

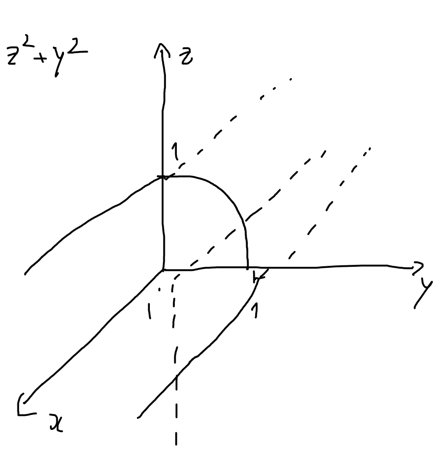
relativamente ao eixo Ox, e cuja densidade de massa é dada por $\sigma(x,y,z) = x(y^2+z^2)$.

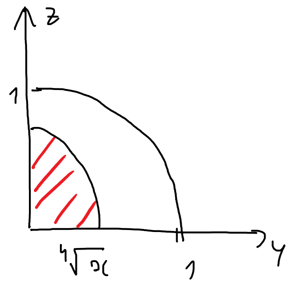
$R^2 \le 1$

$0 \le x \le \sqrt[4]{R^2}$

6) $I_{0_x}(U) = \int_U x\underbrace{(y^2+z^2)}_{R^2} \cdot \underbrace{d^2_{0_x}}_{R^2}$ $\quad R^2 = z^2+y^2$

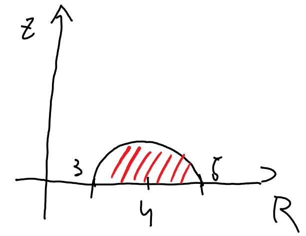
$$V_U = \int_0^{\pi/2}\int_0^1\int_0^{\sqrt[4]{R^2}} dx\, dR\, d\theta$$





$$\int_0^{\pi/2}\int_0^1\int_0^{\sqrt{R}} x(R^2)\cdot R \cdot R^2 \, dx\, dR\, d\theta = \int_0^{\pi/2}\int_0^1 \left[\frac{x^2}{2} R^5\right]_0^{\sqrt{R}} dR\, d\theta = \int_0^{\pi/2}\int_0^1 \frac{R^6}{2} dR\, d\theta = \frac{\pi}{2} \times \frac{1}{14} = \frac{\pi}{28} //$$

b) $\begin{cases} x = R\cos(\theta) \\ y = R\sin(\theta) \\ z = z \end{cases}$ $\quad \begin{cases} 0 \le \theta \le \pi \\ z > 0 \\ (R-4)^2 + z^4 < 1 \end{cases}$



$$\int_0^{\pi}\int_3^5\int_0^{\sqrt{1-(R-4)^2}} 1 \times R \, dz\, dR\, d\theta = \pi \int_3^5 R\sqrt{1-(R-4)^2}\, dR = \pi \int_{-1}^{1} (s+4)\sqrt{1-s^2} \times 1\, ds$$

$s = R-4$

$$= \pi \int_{-1}^{1} s\sqrt{1-s^2}\, ds + \pi \int_{-1}^{1} 4\sqrt{1-s^2}\, ds = 0 + \pi \times 2\pi = 2\pi^2$$

$$\pi \int_{-1}^{1} -\frac{1}{2} \times (-2s) \times \sqrt{1-s^2}\, ds = \left[-\frac{1}{2} \cdot \frac{(1-s^2)^{3/2}}{3/2}\right]_{-1}^{1} = 0$$

$$4\times\pi \int_{-1}^{1} \sqrt{1-s^2}\, ds = 4\times\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2(t)\, dt = 4\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1+\cos(2t)}{2} dt$$

$s = \sin(t)$

$$4\pi \left[\frac{t}{2} + \frac{\sin(2t)}{2}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 2\pi^2$$

7. Calcule o volume de cada uma das regiões:

(a) $A = \{(x,y,z) \in \mathbb{R}^3 : 0 \le x \le 1 - (\sqrt{y^2+z^2} - 1)^2 \,;\, y \ge 0 \,;\, z \ge 0\}$

(b) $B = \{(x,y,z) \in \mathbb{R}^3 : (\sqrt{x^2+y^2} - 4)^2 + z^2 < 1 \,;\, y \ge 0 \,;\, z > 0\}$.

a) $y^2+z^2 = \rho^2$ $\quad \begin{cases} y = \rho\cos(\theta) \\ z = \rho\sin(\theta) \end{cases}$ $\quad (\rho, \theta, x)$ $\quad 1-(\rho-1)^2 \le 0 \iff \rho - 1 \le 1 \iff \rho \le 2$

$x > x$ $\quad 0 < \theta < \frac{\pi}{2}$

$$\int_0^2\int_0^{\frac{\pi}{2}}\int_0^{1-(\rho-1)^2} \rho\, dx\, d\theta\, d\rho = \int_0^2 \frac{\pi}{2}(1-(\rho-1)^2)\rho\, d\rho = \frac{\pi}{2}\int_0^2 (1-\rho^2+2\rho-1)\rho\, d\rho = \frac{\pi}{2}\int_0^2 -\rho^3 + 2\rho^2\, d\rho =$$

$$= \frac{\pi}{2}\left[-\frac{\rho^4}{4} + 2\frac{\rho^3}{3}\right]_0^2 = \frac{\pi}{2} \cdot \frac{4}{3} = \frac{2\pi}{3}$$

8. Calcule $F'(0)$ onde $F : \mathbb{R} \to \mathbb{R}$ é a função definida pela expressão

$$F(t) = \int_0^1 \operatorname{sen}(tx^2 + x^3)dx.$$

8) $\int_0^1 x^2\cos(tx^2+x^3)\, dx \Big|_{t=0} =$

$$\int_0^1 x^2\cos(x^3)\, dx = \left[\frac{\sin(x^3)}{3}\right]_0^1 = \frac{\sin(1)}{3}$$

9. Sendo $V_t = \{(x,y,z) \in \mathbb{R}^3 : 1 \leq x^2+y^2 \leq t \; ; \; 0 \leq z \leq 1 \; ; \; y > 0\}$ e $F\colon [1,+\infty[\to \mathbb{R}$ a função definida por

$$F(t) = \int\int\int_{V_t} \frac{e^{t(x^2+y^2)}}{x^2+y^2} dxdydz,$$

calcule $F'(4)$.

$$F(t) = \int_0^1 \int_0^{\pi} \int_1^{\sqrt{t}} \frac{e^{tR^2}}{R^{\cancel{2}}} \cdot \cancel{R} \, dR \, d\theta \, dz = \pi \int_1^{\sqrt{t}} \frac{e^{tR^2}}{R} dR \qquad R^2 = x^2+y^2$$

$$G(u,v) = \int_1^u \pi \frac{e^{vR^2}}{R} dR$$

$$F'(t) = G'(\sqrt{t}, t) = \underbrace{\frac{\partial G}{\partial u}(\sqrt{t}, t)}_{\pi \cdot \frac{e^{vu^2}}{u}} \frac{1}{2\sqrt{t}} + \frac{\partial G}{\partial v}(\sqrt{t}, t) = \pi \frac{e^{vu^2}}{u} \cdot \frac{1}{2\sqrt{t}} + \int_1^u \pi r e^{vr^2} dr =$$

$$= \pi \frac{e^{t^2}}{2t} + \pi \left(\frac{e^{t^2} - e^{t}}{2t} \right) \xrightarrow[t=4]{} \frac{\pi e^{16}}{8} + \frac{\pi}{8}(e^{16} - e^4) = \frac{\pi(e^{16} + e^{16} - e^4)}{8} = \frac{\pi}{8}(2e^{16} - e^4$$