1. Seja $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ a função definida por

$$f(x,y) = \begin{cases} \dfrac{x^2 y}{3x^2+y^2} & \text{se } (x,y) \neq (0,0) \\ 0 & \text{se } (x,y) = (0,0) \end{cases}.$$

(a) Mostre que f é contínua na origem.

(b) Calcule $\nabla f(0,0)$ e a derivada de f na origem segundo o vector $v = (1,2)$. O que pode concluir acerca da diferenciabilidade de f na origem?

3. Determine e classifique os pontos de estacionaridade da função $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ dada por $f(x,y) = -y + y^2 + x^2 - x^4$.

4. Considere a região $D \subset \mathbb{R}^3$

$$D = \{(x,y,z) \in \mathbb{R}^3 : z \geq \sqrt{x^2+y^2}, z + x^2 + y^2 \leq 2, x \geq 0\}.$$

a) Escreva uma expressão para o volume de D da forma $\int(\int(\int dx)dy)dz$ e da forma $\int(\int(\int dz)dy)dx$.

b) Usando uma mudança de coordenadas apropriada calcule o volume de D.

1) a) $\lim_{x\to 0} \frac{x^2 y}{3x^2+y^2} = 0$

$\lim_{y\to 0} \frac{x^2 y}{3x^2+y^2} = 0$

$0 \leq |f(x,y)| = \frac{|y|\, x^2}{3x^2+y^2} \leq \frac{\sqrt{x^2+y^2}\ \cancel{3x^2+y^2}}{\cancel{3x^2+y^2}}$

$|y| \leq \sqrt{x^2+y^2}$

$x^2 \leq 3x^2+y^2$

$|f(x,y)| \leq \sqrt{x^2+y^2} \to 0$
$(x,y) \to 0$

b) $\frac{\partial f}{\partial x}(0,0) = \frac{d}{dt} f(t,0)\Big|_{t=0} = 0$ $\qquad \frac{\partial f}{\partial t}(t,2t)\Big|_{t=0} = \frac{\partial}{\partial t} \frac{t^2 \cdot 2t}{3t^2+4t^2} = \frac{2}{7}$

$\frac{\partial f}{\partial y}(0,0) = \frac{d}{dt} f(0,t)\Big|_{t=0} = 0$

2. Seja $g : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ uma função de classe C^1 tal que

$$Dg(1,-1) = \begin{bmatrix} 0 & 1 \\ 2 & 0 \\ 0 & 3 \end{bmatrix},$$

e seja $h(x,y) = (\cos(x+y^2), xy)$. Calcule $D(g\circ h)(-1,1)$.

$Df\,o_g = 0$

$Df(g)\circ Dg$

$D(g\circ h)(-1,1) = Dg(h(-1,1)) \cdot Dh(-1,1)$

$$\begin{bmatrix} 0 & 1 \\ 2 & 0 \\ 0 & 3 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 3 & -3 \end{bmatrix}$$

3 $\nabla f(x,y) = (2x - 4x^3, -1+2y)$ $\quad \begin{cases} 2x - 4x^3 = 0 \\ -1+2y = 0 \end{cases} \Leftrightarrow \begin{cases} x = 0 \vee 2 - 4x^2 = 0 \\ y = \frac{1}{2} \end{cases} \Leftrightarrow \begin{cases} x = 0 \vee x = \frac{\sqrt{2}}{2} \vee x = -\frac{\sqrt{2}}{2} \\ y = \frac{1}{2} \end{cases}$

$(0,\frac{1}{2})$ $(\frac{\sqrt{2}}{2},\frac{1}{2})$ $(-\frac{\sqrt{2}}{2},\frac{1}{2})$

$f(0,\frac{1}{2}) = -\frac{1}{2} + \frac{1}{4} = -\frac{1}{4}$ $\qquad f(-\frac{\sqrt{2}}{2},\frac{1}{2}) = 0$

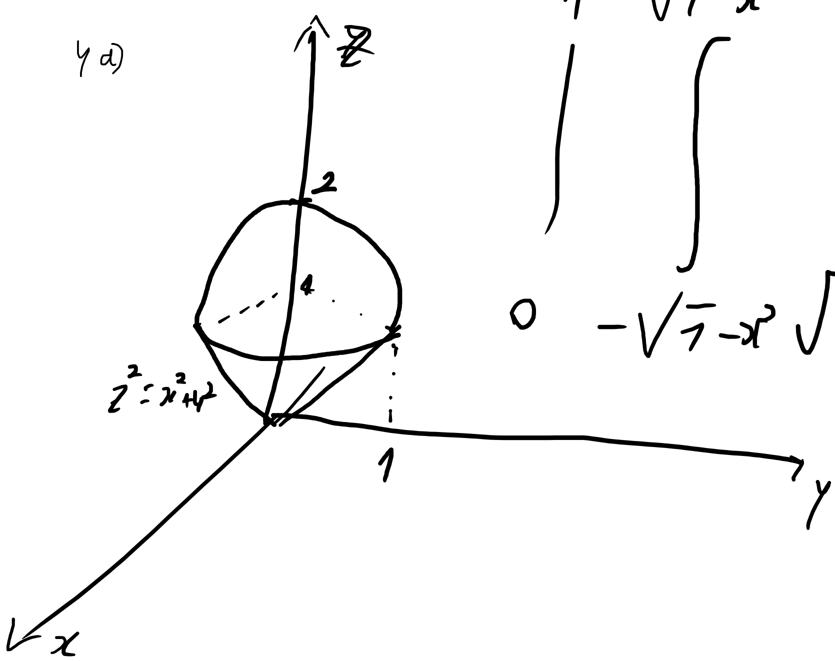
$f(\frac{\sqrt{2}}{2},\frac{1}{2}) = -\frac{1}{4} + \frac{1}{2} - \frac{1}{4} = 0$

$(0,\frac{1}{2})$ Mim

$-\frac{\sqrt{2}}{2},\frac{1}{2}$ Sela

$\frac{\sqrt{2}}{2},\frac{1}{2}$ Sela

4 a)



$$\int_0^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{2-x^2-y^2} 1\, dz\, dy\, dx$$

b) Mga $x^2+y^2 = \rho^2$ $\qquad z \geq \rho$

$z = z$ $\qquad z \leq 2-\rho^2$

$x = \rho\cos(\theta)$ $\qquad \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$y = \rho\sin(\theta)$

$$\int_0^1 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{\rho}^{2-\rho^2} \rho\, dz\, d\theta\, d\rho = \pi \int_0^1 2\rho - \rho^3 - \rho^2\, d\rho = \pi \left[\rho^2 - \frac{\rho^4}{4} - \frac{\rho^3}{3}\right]_0^1$$

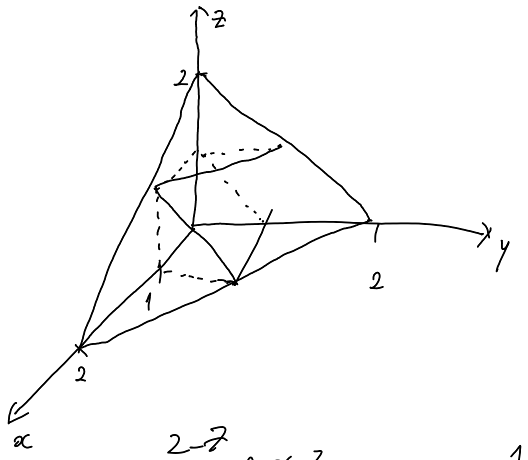
5. Considere o conjunto

$$V = \{(x,y,z)\in\mathbb{R}^3 : x+y+z<2\,;\ 0<x<1\,;\ y>0\,;\ z>0\}.$$

(a) Escreva expressões para o volume de V em termos de integrais iterados das duas formas seguintes: $\int(\int(\int dz)dy)dx$; $\int(\int(\int dx)dy)dz$.

(b) Calcule o volume de V.

a)



$x+y+z<2$
$0<x<1$
$y>0$
$z>0$

$$\int_0^1\int_0^{2-x}\int_0^{2-x-y} 1\,dz\,dy\,dx$$

$$\int_0^{2-x} 2-x-y\,dy$$

$$\left[2y-xy-\frac{y^2}{2}\right]^{2-x}$$

$$2(2-x)-x(2-x)-\frac{(2-x)^2}{2}$$

$$\left[4-2x-2x+x^2-2+2x-\frac{x^2}{2}\right]_0^1$$

$$\left[2-2x+\frac{x^2}{2}\,dx\right]_0^1$$

$$\left[2x-x^2+\frac{x^3}{6}\right]_0^1 \quad 2-1+\frac{1}{6}=\frac{7}{6}$$

$$\int_0^1\int_0^{\cancel{2-x}\ 1-z}\int_0^1 1\,dx\,dy\,dz + \int_1^2\int_0^{\cancel{2-x}\ 2-z}\int_0^{2-x-z} 1\,dx\,dy + \int_0^1\int_{\cancel{2-x}\ 1-z}^{\cancel{2-x}\ 2-z}\int_0^{2-x-z} \quad ✓$$

1. Considere a função

$$f(x,y)=\begin{cases}5+\dfrac{x^2y^4}{(x^2+y^2)^2}, & \text{se } (x,y)\neq(0,0)\\ a, & \text{se } (x,y)=(0,0).\end{cases}$$

Determine o valor de a para que f seja contínua na origem.

$$0\le|f(x,y)|=5+\frac{x^2y^4}{(x^2+y^2)^2}\le 5+\frac{y^4}{(x^2+y^2)}\le$$

$$5+y^2$$

$x^2\le x^2+y^2$

$y^2\le x^2+y^2$

$$\lim_{x,y\to 0,0} f(x,y)=5$$

logo $a=5$

2. Considere a função $g:\mathbb{R}^2\to\mathbb{R}^2$ definida por $g(x,y)=(xy,x^2+y^2)$ e a função $f:\mathbb{R}^3\to\mathbb{R}^2$, de classe C^1, e tal que $f(1,1,1)=(1,2)$ e

$$Df(1,1,1)=\begin{bmatrix}1&0&1\\0&1&1\end{bmatrix}.$$

Calcule a derivada da função $g\circ f$ no ponto $(1,1,1)$, segundo o vector $v=(\frac{1}{\sqrt2},0,\frac{1}{\sqrt2})$.

2

$$D(g\circ f)=Dg(f)\cdot Df$$

$$Dg=\begin{bmatrix}y&x\\2x&2y\end{bmatrix}$$

$$Dg(f(a))=\begin{bmatrix}2&1\\2&4\end{bmatrix}\cdot\begin{bmatrix}1&0&1\\0&1&1\end{bmatrix}=\begin{matrix}2&1&3\\2&4&6\end{matrix}$$

$2\times2 \quad 2\times3$

$$\begin{pmatrix}2&1&3\\2&4&6\end{pmatrix}\begin{bmatrix}\frac{1}{\sqrt2}\\0\\\frac{1}{\sqrt2}\end{bmatrix}$$

$$\frac{2}{\sqrt2}+\frac{3}{\sqrt2}\ ,\ \frac{2}{\sqrt2}+\frac{6}{\sqrt2}$$

$$\left(\frac{5}{\sqrt2},\frac{8}{\sqrt2}\right)$$

3. Determine e classifique os pontos críticos do campo escalar dado por

$$f(x,y,z) = 3x^2 + 2\sqrt{2}xy + 2y^2 + \frac{2}{3}z^3 - 2z.$$

$$\nabla f(x,y,z) = \left(6x + 2\sqrt{2}y,\ 2\sqrt{2}x + 4y,\ 2z^2 - 2\right)$$

$$\lambda = \frac{10 \pm \sqrt{100 - 4 \times 16}}{2}$$

$$\begin{cases} 6x + 2\sqrt{3}y = 0 \\ 2\sqrt{2}x + y = 0 \\ 2z^2 - 2 = 0 \end{cases} \Leftrightarrow \begin{cases} 6x + 2\sqrt{2}(-2\sqrt{2})x = 0 \\ y = -2\sqrt{2}x \\ z = 1 \vee z = -1 \end{cases} \Leftrightarrow \begin{cases} 6x - 8x = 0 \end{cases} = \begin{array}{l} x = 0 \\ y = 0 \\ z = 1 \end{array} \quad \begin{array}{l} x = 0 \\ y = 0 \\ \cancel{y} = -1 \end{array}$$

$$\lambda^2 - 10\lambda + 16 = 0 \qquad \frac{10 \pm 6}{2} > 0$$

$(0,0,1)$ $\quad f(0,0,1) = \frac{2}{3} - 2 = -\frac{4}{3}$ $\quad f(0,0,-1) = -\frac{2}{3} + 4 = \frac{10}{3}$

$(0,0,-1)$

$$\begin{bmatrix} 6 & 2\sqrt{2} & 0 \\ 2\sqrt{2} & 4 & 0 \\ 0 & 0 & 4z \end{bmatrix}$$

$$(6-\lambda)(4-\lambda) = 8$$

$$24 - 6\lambda - 4\lambda + \lambda^2 = 8$$

$$-10\lambda + \lambda^2 + 16 = 0$$

$$\begin{vmatrix} 6-\lambda & 2\sqrt{2} & 0 \\ 2\sqrt{2} & 4-\lambda & 0 \\ 0 & 0 & -4-\lambda \end{vmatrix}$$

$\lambda = 4$ Def Positiva

$\lambda = -4$ ~~Def~~ Indef

$$(6-\lambda)(4-\lambda)(-4-\lambda) - 8(4-\lambda) = 0 \Leftrightarrow (-4-\lambda)\left((6-\lambda)(4-\lambda) - 8\right) = 0$$

4. Calcule o trabalho do campo vectorial

$$F(x,y) = \left(-\frac{2y}{(x+1)^2+y^2} - y^3\,,\ \frac{2x+2}{(x+1)^2+y^2} + x^3 + y\right),$$

ao longo da circunferência de raio igual a dois e centro na origem de $\mathbb{R}^2$ e no sentido positivo.

$\int F\, dg$ $\qquad g(\theta) = (2\cos(\theta), 2\sin(\theta))$

$F(x,y) = \underbrace{\left(-\frac{2y}{(x+1)^2+y^2}, \frac{2(x+1)}{(x+1)^2+y^2}\right)}_{-4\pi} + (-y^3, x^3+y) = 32\pi - 4\pi = 28\pi$

$\int F(g)\cdot g' = \int_0^{2\pi} (-8\sin^3(\theta), 8\cos^3(\theta) + 2\sin(\theta)) \cdot (-2\sin(\theta), 2\cos(\theta))$

$16\sin^4(\theta) + 16\cos^4(\theta) + \underbrace{4\sin(\theta)\cos(\theta)}_{0}$ $\qquad 16\cdot 2\pi = 32\pi$

5. Considere a superfície

$$S = \{(x,y,z) \in \mathbb{R}^3 : 1 + x^2 = y^2 + z^2\,;\ 0 < x < 2\}.$$

orientada com a normal ν que no ponto $(1, 0, \sqrt{2})$ tem terceira componente positiva.

a) Calcule o fluxo do campo vectorial $F(x,y,z) = (x, y, -z)$ através de S no sentido da normal ν, usando o teorema da divergência.

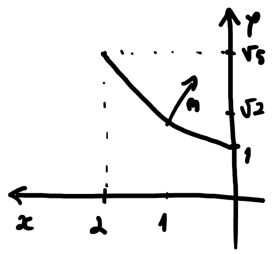
b) Calcule o fluxo do campo vectorial $G(x,y,z) = (-x, -y, 2z)$ através de S no sentido da normal ν, usando o teorema de Stokes.

c) Calcule o fluxo do rotacional do campo $H(x,y,z) = (-y, x, z)$ através de S no sentido da normal ν.

5) $\quad 1+x^2 = y^2+z^2 \quad 0<x<2$

$y^2+z^2 = r^2$

$1+x^2 = r^2$



a) $\int_S F\cdot n + \int_A F\cdot n = \int_S \operatorname{div} F \iff \int_S 1+1-1 - \int_A (x,y,-z)\cdot(1,0,0) - \int_B (x,y,-z)\cdot(-1,0,0)$

$\int_S 1 - \int_A x + \int_B x = \int_0^2\int_0^{\sqrt{1+x^2}}\int_0^{2\pi} r\, d\theta\, dr\, dx - \underbrace{\int_A 2}_{\text{Área do círculo de raio } \sqrt5} + \underbrace{\int_B 0}_{\text{Área do círculo de raio 1}} = \int_0^2\int_0^{\sqrt{1+x^2}}\int_0^{2\pi} r\, d\theta\, dr\, dx - 2\cdot\pi\cdot 5 = \frac{14\pi}{3} - 10\pi = -\frac{16}{3}\pi$

$2\pi\int_0^2 \frac{1+x^2}{2}\, dx = 2\pi\int_0^2 \frac12 + \frac{x^2}{2}\, dx = 2\pi\left[\frac{x}{2} + \frac{x^3}{6}\right]_0^2 = \frac73\cdot 2\pi = \frac{14}{3}\pi$

b) $\int_S G\cdot n = \int_S \operatorname{Rot}(f)\cdot n = \oint_{\partial S} f$

$\operatorname{Rot}(f) = G$

$\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z} = -x \rightsquigarrow x - 2x = -x$

$\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x} = -y \rightsquigarrow f_3 = yx$

$\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} = 2z \rightsquigarrow f_2 = 2zx$

$f: (0, 2zx, yx)$ $\qquad \oint_{\partial S} f = -\int f(g_B)\cdot g_B' + \overbrace{\int f(g_A)\cdot g_A'}^{0} =$

$= 10\pi$ //

$g_B(\theta) = (2, \sqrt5\cos(\theta), \sqrt5\sin(\theta))$

$g_A(\theta) = (0, \cos(\theta), \sin(\theta))$

$\int_0^{2\pi} f(g_B)\cdot g_B' = \int_0^{2\pi} (0, 2\cdot 2\cdot\sqrt5\sin(\theta), 2\sqrt5\cos(\theta))\cdot(0, -\sin(\theta)\sqrt5, \cos(\theta)\sqrt5) = -10\pi$

c) $\int \operatorname{Rot}(H)\cdot n = \int_{\partial S} H = 0 + 0$ $\qquad H(x,y,z) = (-y, x, z)$

$\int_0^{2\pi} H(g_B)\cdot g_B' = \int_0^{2\pi} (-\sqrt5\cos(\theta), 2, \sqrt5\sin(\theta))\cdot(0, -\sin(\theta)\sqrt5, \sqrt5\cos(\theta)) =$

$\int_0^{2\pi} 0 - 2\sqrt5\sin(\theta) + 5\sin(\theta)\cos(\theta)\, d\theta = 0$

$\int_0^{2\pi} H(g_A)\cdot g_A' = \int_0^{2\pi} (-\cos(\theta), 0, \sin(\theta))\cdot(0, -\sin(\theta), \cos(\theta)) = \int_0^{2\pi} \sin(\theta)\cos(\theta) = 0$

6. Calcule, em coordenadas cilíndricas, o momento de inércia relativo ao eixo Oz do sólido

$$V = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2<1\,;\, 0<z<1+x^2+y^2\,;\, y>0\},$$

cuja densidade de massa é constante e igual a um.

$x^2+y^2<1$ $\quad$ $x^2+y^2=\rho^2$ $\quad$ $\rho \in [0,1]$

$0<z<1+x^2+y^2$ $\quad$ $0<z<1+\rho^2$ $\quad$ $z \in [0, 1+\rho^2]$

$y>0$ $\quad$ $Y>0$ $\quad$ $\theta \in [\pi, 2\pi]$

$$\int_0^1 \int_0^{(1+\rho^2)} \int_\pi^{2\pi} \rho^3 \, d\theta \, dz \, d\rho$$

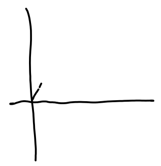
$$\int_0 1 \cdot d^2_{Oz}$$

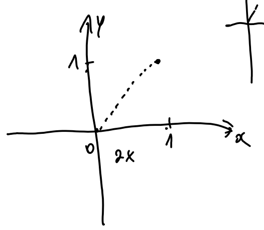
$$\pi \int_0^1 \rho^3(1+\rho^2)\, d\rho$$

$\rho^3+\rho^5$

$\frac{\rho^4}{4}+\frac{\rho^6}{6}$ $\quad$ $\frac{1}{4}+\frac{1}{6}$

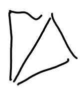
$f(x,y)=\max(x,y)$ $\quad$ $f(x,y)=\begin{cases} y\le x\,;\, f(x,y)=x \\ x<y\,;\, f(x,y)=y \end{cases}$





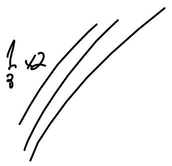
$$\int_0^1\int_0^1 f(x,y)\,dx\,dy$$

$$2\times\int_0^1\int_0^x x\,dy\,dx$$



$$2\times\int_0^1 x^2\,dx \qquad \frac{x^3}{3}$$

$\frac{1}{3}\times 2$



1. Considere o conjunto

$$L = \{(x,y,z) \in \mathbb{R}^3 : e^x+e^y+z=3;\ x-z+1=0\}.$$

(a) Mostre que L é uma variedade e indique a sua dimensão.

(b) Mostre que L é o gráfico de uma função $f(y)=(x(y),z(y))$, de classe C^1, numa vizinhança do ponto $(0,0,1)$ e calcule $x'(0)$.

a) $g(x) = (x, \ln(e^x - x - 2), x+1)$ $\quad$ Dim 1

b) $\begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \end{bmatrix}$

$$f(y) = \begin{cases} e^{g_1(y)} + e^y + g_2(y) = 3 \\ g_1(y) - g_2 + 1 = 0 \end{cases}$$

$g_1'(0) + 1 + g_1'(y) = 0$

$g_1'(0) = -\frac{1}{2}$

$$f'(0) = \begin{cases} \frac{d}{dy} e^{g_1(y)} + e^y + g_2'(y) = 0 \\ g_1'(y) - g_2'(z) = 0 \end{cases} \iff \begin{cases} g_1'(y) e^{g_1(y)} + e^y + g_2'(y) = 0 \\ g_1'(y) = g_2'(y) \end{cases}$$

2. Determine o vector de $\mathbb{R}^3$ cujo comprimento é igual a 3 e cuja soma das componentes é a maior possível.

$$3 = \sqrt{V_x^2 + V_y^2 + V_z^2} \qquad \sqrt{3}$$

$$V_x + V_y + V_z = 3$$

3. Determine um ponto da superfície $z = 2 - xy$ para o qual a recta normal à superfície nesse ponto passa na origem.

$g(x,y) = (x, y, 2-xy)$ $\quad (1,0,-y)\alpha + (0,1,-x)\beta \quad \alpha, \beta \in \mathbb{R}$

$$Jg = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -y & -x \end{bmatrix}$$

$(1,0,-y)\cdot(a,b,c) = 0 \quad a - yc = 0$

$(0,1,-x)\cdot(a,b,c) = 0 \quad b - cx = 0$

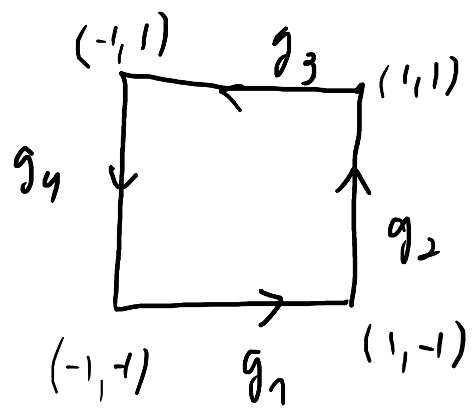
$(y, x, 1) \quad \in \mathbb{R}$

$$\begin{cases} (0,0,0) - (y,x,1)\lambda + (x,y,z) \\ z = 2 - xy \end{cases} \quad \begin{cases} 0 = y\lambda + x \\ 0 = x\lambda + y \\ 0 = \lambda + z \\ z = 2 - xy \end{cases} \Leftrightarrow \begin{cases} \lambda = -\frac{x}{y} \\ \lambda = -\frac{y}{x} \\ z = 1 \\ z = 2 - x^2 \end{cases} \quad x = y$$

$$\boxed{\begin{matrix} (1,1,1) \\ (-1,-1,1) \end{matrix}}$$

4. Considere o campo vectorial $G(x,y) = (-y + x^3, x + y^5) + \nabla\psi(x,y)$ onde $\psi(x,y) = \cos\left(\frac{\pi x y}{2}\right)$. Calcule o trabalho de G ao longo da fronteira do quadrado $\{(x,y) \in \mathbb{R}^2 : |x| \leq 1; |y| \leq 1\}$ percorrida no sentido anti-horário.

$f(x,y) = (-y + x^3, x + y^5)$



$\int \nabla\psi = 0$

$g_1(t) = (t, -1) \quad t \in [-1, 1]$

$g_2(t) = (1, t) \quad t \in [-1, 1]$

$g_3(t) = (t, 1) \quad t \in [1, -1]$

$g_4(t) = (-1, t) \quad t \in [1, -1]$

$$\int f\, dg_1 = \int_{-1}^{1} f(g_1)\cdot(1,0)\, dt = \int_{-1}^{1} 1 + t^3\, dt = 2$$

$$\int_{1}^{-1} f(g_3)\cdot(1,0)\, dt = \int_{1}^{-1} -1 + t^3 = 2$$

$$\int f\, dg_2 = \int_{-1}^{1} f(g_2)\cdot(0,1)\, dt = \int_{-1}^{1} 1 + t^5\, dt = 2$$

$$\int_{1}^{-1} f(g_4)\cdot(0,1) = \int_{1}^{-1} -1 + t^5 = 0 \quad \Big| \to 8$$

$$\int_{-1}^{1}\int_{-1}^{1} (1+1)\, dx\, dy = \int_{-1}^{1} 4\, dy = 8$$

5. Sejam $F(x,y,z) = (x,y,z+1)$,

$$S = \{(x,y,z)\in\mathbb{R}^3 : z = 1-x^2-y^2;\ y>0;\ z>0\}$$

e seja $n=(n_1,n_2,n_3)$ a normal a S, unitária, tal que $n_3>0$.

(a) Calcule a área de S.

(b) Calcule o fluxo $\iint_S F\cdot n$ pela definição.

(c) Calcule o fluxo $\iint_S F\cdot n$ usando o Teorema da Divergência.

c)

$z = 1-x^2-y^2$

$y>0$

$z>0$

Seja $x^2+y^2=\rho^2$

$z = 1-\rho^2$

$x = \rho\cos(\theta)$

$y = \rho\sin(\theta)$

$z\in[0;1]$

$\theta\in[-\frac{\pi}{2};\frac{\pi}{2}]$

$\rho\in[0;1]$

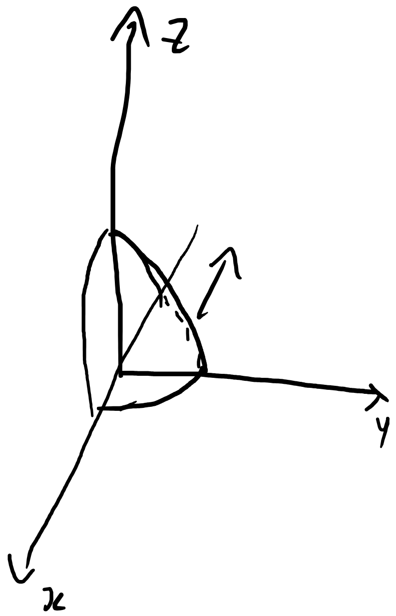
$$\int_S F\cdot n = \int_S \operatorname{div} F - \int_A F\cdot n$$

$$g(\theta,\rho) = (\rho\cos(\theta), \rho\sin(\theta), 1-\rho^2)$$

$$Dg(\theta,\rho) = \begin{bmatrix} -\rho\sin(\theta) & \cos(\theta) \\ \rho\cos(\theta) & \sin(\theta) \\ 0 & -2\rho \end{bmatrix}$$

$$\left\|\frac{\partial g}{\partial\rho}\times\frac{\partial g}{\partial\theta}\right\| = \left\|\begin{vmatrix} e_1 & e_2 & e_3 \\ -\rho\sin(\theta) & \rho\cos(\theta) & 0 \\ \cos(\theta) & \sin(\theta) & -2\rho \end{vmatrix}\right\| =$$

$$\sqrt{(-2\rho^2\cos(\theta))^2+(-2\rho^2\sin(\theta))^2+(-\rho)^2} = \sqrt{4\rho^4+\rho^2} = \rho\sqrt{4\rho^2+1}$$



$$\pi\int_0^1 \sqrt{4\rho^4+\rho^2}\,d\rho = \frac{\pi}{12}\left(5^{3/2}-1\right)$$

b) $$\int_0^\pi\int_0^1 (\rho\cos(\theta), \rho\sin(\theta), 2-\rho^2)\cdot(-2\rho^2\cos(\theta), -2\rho^2\sin(\theta), -\rho) = \int_0^\pi\int_0^1 -2\rho^3-(2-\rho^2)\rho\,d\rho = \pi\int_0^1 -\rho^3-2\rho = \frac{5}{4}\pi$$

c) $$\int \operatorname{div} F = \int_S F\cdot n + \int_A F\cdot n$$

$$3\int_S = 3\int_0^\pi\int_0^1\int_0^{1-\rho^2}\rho\,dz\,d\rho\,d\theta = 3\pi\int_0^1 \rho-\rho^3\,d\rho = 3\pi\left[\frac{\rho^2}{2}-\frac{\rho^4}{4}\right]_0^1 = 3\pi\left(\frac{1}{2}-\frac{1}{4}\right) =$$

1. Considere o sistema

$$\begin{cases} 2ye^{y-1} + t = 3 \\ x^2 + y^2 = \cos^2(\frac{\pi}{2}t) + 1. \end{cases}$$

a) Mostre que este sistema define, numa vizinhança do ponto $(t_0, x_0, y_0) = (2, 1, 1)$, as variáveis x e y como funções de t, de classe C^1.

b) Calcule $x'(2)$ e $y'(2)$.

a) $u(t,x,y) = 2ye^{y-1} + t - 3$

$v(t,x,y) = x^2 + y^2 - \cos^2(\frac{\pi}{2}t) + 1$

$$Df(u,v) = \begin{bmatrix} 1 & 0 & 2ye^{y-1} + 2e^{y-1} \\ \pi\cos(\frac{\pi}{2}t)\sin(\frac{\pi}{2}t) & 2x & 2y \end{bmatrix} \xrightarrow{2,1,1} \begin{array}{c} \begin{array}{ccc} t & x & y \end{array} \\ \begin{bmatrix} 1 & 0 & 4 \\ 0 & 2 & 2 \end{bmatrix} \end{array}$$

$$\begin{array}{cc} x & y \end{array}$$
$$\begin{vmatrix} 0 & 4 \\ 2 & 2 \end{vmatrix} \neq 0 \rightarrow \text{logo pelo TFImp} \begin{cases} x = x(t) \\ y = y(t) \\ t = t \end{cases}$$

$$f(t) = \begin{cases} 2 \cdot y(t) \cdot e^{y(t)-1} + t = 3 \\ (x(t))^2 + (y(t))^2 = \cos^2(\frac{\pi}{2}t) + 1 \end{cases}$$

$$y'(t) = \begin{cases} 2 \cdot y(t) \cdot y'(t) \cdot e^{y(t)-1} + 2 \cdot y'(t) \cdot e^{y(t)-1} + 1 = 0 \\ 2 \cdot x(t) \cdot x'(t) + 2 \cdot y(t) \cdot y'(t) = \pi\cos(\frac{\pi}{2}t)\sin(\frac{\pi}{2}t) \end{cases}$$

$$f'(2) = \begin{cases} 2y'(2) + 2y'(2) = -1 \\ 2x'(2) + 2y'(2) = 0 \end{cases} \Leftrightarrow \begin{cases} y'(2) = -\frac{1}{4} \\ x'(2) = \frac{1}{4} \end{cases}$$

2. Considere o conjunto

$$M = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1 \; ; \; x = y \; ; \; z > 0\}.$$

a) Mostre que M é uma variedade e determine a respectiva dimensão.

b) Determine um vector normal a M no ponto $(\frac{1}{2}, \frac{1}{2}, \frac{1}{\sqrt{2}})$.

a) $\begin{array}{l} x^2+y^2+z^2=1 \\ x=y \\ z>0 \end{array}$ $\quad \begin{array}{l} x^2+y^2+z^2-1=0 \\ x-y=0 \end{array}$ $\quad Df = \begin{bmatrix} 2x & 2y & 2z \\ 1 & -1 & 0 \end{bmatrix}$ car 2

Dim $M = 1$

b) $(4x, 0, 2z) \times \quad (2, 0, \frac{2}{\sqrt{2}})$

$(1,0,0) + (0,0,-\sqrt{2}) = (1,0,-\sqrt{2})$

3. Determine o valor mínimo da função $f(x,y,z) = xy + z^2$ no conjunto definido por

$$\begin{cases} x^2 + y^2 + z^2 = 1 \\ y = x. \end{cases}$$

$$\begin{cases} x^2+y^2+z^2-1=0 \\ y-x=0 \\ (y, x, 2z) = \lambda(2x, 2y, 2z) + \delta(-1,1,0) \end{cases} \Leftrightarrow \begin{cases} x^2+y^2+z^2-1=0 \\ y-x=0 \\ y = 2\lambda x - \delta \\ x = 2\lambda y + \delta \\ z = \lambda z \end{cases} \Leftrightarrow \begin{cases} 2x^2+z^2-1=0 \\ x = 2\lambda x - \delta \\ x = 2\lambda x + \delta \\ z = \lambda z \end{cases}$$

$\delta = 0$

$$\Leftrightarrow \begin{cases} 2x^2+z^2-1=0 \\ x = 2\lambda x \\ z = \lambda z \end{cases} \Leftrightarrow \begin{cases} 2x^2 = 1 \Leftrightarrow x = \pm\sqrt{\frac{1}{2}} = \pm\frac{\sqrt{2}}{2} \\ x = x \\ z = 0 \\ \lambda = \frac{1}{2} \end{cases}$$

$(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0) \quad f = \frac{\sqrt{2} \cdot \sqrt{2}}{2 \cdot 2} = \frac{1}{2}$

$(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 0) \quad f = \frac{\sqrt{2} \cdot \sqrt{2}}{2 \cdot 2} = \frac{1}{2}$

1. Mostre que a equação $y^5 + 2xy^3 + yz = 1$ define y como função de x e z, de classe C^1, em alguma vizinhança do ponto $(0, 1, 0)$ e calcule as derivadas $\frac{\partial y}{\partial x}(0,0)$ e $\frac{\partial y}{\partial z}(0,0)$.

2. Mostre que o conjunto $M = \{(x, y, z) \in \mathbb{R}^3 : y^5 + 2xy^3 + yz = 1\}$ é uma variedade, indicando a respectiva dimensão, e determine uma base do espaço tangente a M no ponto $(0, 1, 0)$.

1) seja $h(x,y,z) = y^5 + 2xy^3 + yz - 1$

$$Dh = \begin{array}{ccc} x & y & z \\ \left[2y^3\right. & 5y^4 + 6xy^2 + z & \left.y\right] \end{array} \Rightarrow \begin{array}{ccc} x & y & z \\ \left[2\right. & 5 & \left.1\right] \end{array}$$

Seja $\varphi(x,z) = \varphi^5 + 2x\varphi^3 + \varphi z - 1$ $\qquad (0,1,0)$

$$\frac{\partial \varphi}{\partial x} = \frac{\partial \varphi^5}{\partial x} + 2\cdot\left(\varphi^3 + x\frac{\partial \varphi^3}{\partial x}\right) + z\cdot\frac{\partial \varphi}{\partial x} = 0$$

$$\frac{\partial \varphi}{\partial z}: \quad \frac{\partial \varphi^5}{\partial z} + 2x\cdot\frac{\partial \varphi^3}{\partial z} + \varphi + \frac{\partial \varphi}{\partial x} z = 0$$

$$5\cdot\frac{\partial \varphi}{\partial x} + 2 = 0 \qquad \frac{\partial \varphi}{\partial x} = -\frac{2}{5}$$

$$\Leftrightarrow \frac{\partial \varphi}{\partial z} = -\frac{1}{5}$$

$$5\cdot\frac{\partial \varphi}{\partial z} + 1 = 0$$

2) $f(x,y,z) = y^5 + 2xy^3 + yz - 1$ $\quad Df = [2y^3 \quad 5y^4 + 6xy^2 + z \quad y]$

car 0

logo Var-2

2 5 1

$(0, -5, 1)$ e $(1, 2, 0)$

3. Determine o mínimo da função $f : \mathbb{R}^3 \to \mathbb{R}$ definida por $f(x, y, z) = y$ no conjunto dado pelas equações $x^2 + z^2 = 1$, $y + z = 1$.

4. Sejam $F : \mathbb{R}^2 \to \mathbb{R}^2$ e $G : \mathbb{R}^2 \setminus \{(-1, -1)\} \to \mathbb{R}^2$ os campos vetoriais dados por

$$F(x, y) = (3x^2y^2, 2x^3y + y),$$

$$G(x, y) = \left(\frac{y + 1}{(x + 1)^2 + (y + 1)^2}, -\frac{x + 1}{(x + 1)^2 + (y + 1)^2}\right).$$

Determine, justificando, se os seguintes campos vetoriais são conservativos:

a) $F(x, y)$

b) $G(x, y) - F(x, y)$

3 $\begin{cases} x^2 + z^2 = 1 \\ y + z = 1 \\ (0,1,0) = \lambda_1(2x, 0, 2z) + \lambda_2(0, 1, 1) \end{cases}$

$$\begin{cases} x^2 + z^2 - 1 = 0 \\ y + z - 1 = 0 \\ 0 = \lambda_1 2x \\ 0 = \lambda_1 2z + \lambda_2 \end{cases} \Leftrightarrow \begin{cases} = \\ = \\ \lambda_1 = 0 \vee x = 0 \\ 1 = \lambda_2 \\ 0 = \lambda_1 2z + 1 \end{cases} \Leftrightarrow \begin{cases} \\ \end{cases}$$

$x = 0$

$\begin{cases} z = 1 \vee z = -1 \\ y = 0 \vee y = -2 \\ x = 0 \vee x = 0 \end{cases}$ $\qquad \lambda_1 = 0$ IMP

$f = -2$ //

$(0,0,1)$

$(0,-2,-1)$

4. $\frac{\partial f_2}{\partial x} = 6x^2y$

$\checkmark$ é fechado

$\frac{\partial f_1}{\partial y} = 6x^2y$

$$\frac{\partial \phi}{\partial x} = 3x^2y^2 \rightsquigarrow \phi = x^3y^2 + \alpha(y)$$

$$\alpha(y) = \frac{y^2}{2}$$

$x^3y^2 + \frac{y^2}{2}$ conservado

5. Considere a superfície $S = \{(x,y,z) \in \mathbb{R}^3 : z = 1 - \sqrt{x^2+y^2},\ \alpha < z < 1\}$, em que $\alpha \in \mathbb{R}$ e $0 < \alpha < 1$. Parametrize S e calcule a respectiva área em função de α.

6. Seja $F : \mathbb{R}^3 \to \mathbb{R}^3$ um campo vetorial de classe C^1 tal que $F(x,y,0) = (3y, -3x, \operatorname{sen}(x^2+y^2))$ e seja $P = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 = 2,\ z > 0\}$. Calcule o fluxo $\int_P \operatorname{rot} F \cdot n$, onde n é uma normal unitária a P tal que $n_3 > 0$.

$$D g = \begin{bmatrix} \cos(\theta) & -\rho\, \operatorname{sen}(\theta) \\ \operatorname{sen}(\theta) & \rho \cos(\theta) \\ -1 & 0 \end{bmatrix}$$

5) $x^2+y^2=\rho^2$ $\quad g(\theta,\rho) = (\rho\cos, \rho\operatorname{sen}, 1-\rho)$ $\quad \| D_\rho \theta \times D_\theta g \| =$

$z = 1-\rho$

$$\int_0^{2\pi}\int_\alpha^1 \rho\, d\rho\, d\theta = \frac{\rho^2}{2}$$

$$\left| \begin{matrix} e_1 & e_2 & e_3 \\ \cos & \operatorname{sen} & -1 \\ -\rho\operatorname{sen} & \rho\cos & 0 \end{matrix} \right| = \left| (-\rho\cos(\theta), \rho\operatorname{sen}(\theta), \rho \right|$$

$$\sqrt{\rho^2+\rho^2}$$

$$\sqrt{2}\rho$$

$$2\pi \frac{1-\alpha^2}{2} = \pi(1-\alpha^2)\sqrt{2}$$

6.)

$$\int_P \operatorname{Rot}(F) \cdot n = \oint_{\partial P} F$$

$$\int_0^{2\pi} F(\sqrt{2}\cos(\theta), \sqrt{2}\operatorname{sen}(\theta), 0) \cdot (-\sqrt{2}\operatorname{sen}(\theta), \sqrt{2}\cos(\theta), 0)$$

$$(3\sqrt{2}\operatorname{sen}(\theta), -3\sqrt{2}\cos(\theta), ?) \cdot (-\sqrt{2}\operatorname{sen}(\theta), \sqrt{2}\cos(\theta), 0) =$$

$$\int_0^{2\pi} 3 \cdot 2 \qquad 12\pi$$

7. Considere a superfície

$$S = \{(x,y,z) \in \mathbb{R}^3 : y^2+z^2 = 1,\ 0 < x < 2\}$$

e o campo vetorial $F : \mathbb{R}^3 \to \mathbb{R}^3$ dado por $F(x,y,z) = (x^2 - x, \operatorname{sen}(x^2) - 2xy, \cos(x^2) + z)$. Calcule o fluxo $\int_S F \cdot n$, onde n é uma normal unitária a S tal que $n(1,1,0) = (0,1,0)$.

8. Seja $V \subset \mathbb{R}^3$ um domínio regular, $\phi : \mathbb{R}^3 \to \mathbb{R}$ um campo escalar de classe C^1 e $F : \mathbb{R}^3 \to \mathbb{R}^3$ um campo vectorial de classe C^1 que se anula em ∂V. Mostre que se tem

$$\int_V \nabla\phi \cdot \operatorname{rot} F = 0.$$

8

$$\int_V \nabla\phi \cdot \operatorname{Rot} F = \int_V \operatorname{div}(\phi \operatorname{Rot} F) =$$

$$\int_{\partial V} \phi \operatorname{Rot} F \cdot n = 0$$

7.

$$\int_S F\cdot n + \int_{TA} F\cdot n + \int_{TB} F \cdot n = \int_S \operatorname{div} F \iff \int_S F\cdot n = -\int_{TA} F \cdot n_{ext} - \int_{TB} F \cdot n_{ext}$$

$$\operatorname{div} F = 2x - 1 - 2x + 1 = 0$$

$x=0$

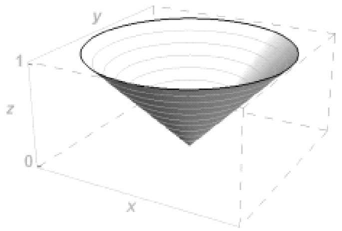
$$-\int_A F \cdot (1,0,0) = 0$$

$x=2$

$$2\operatorname{vol}(B) = 2 \cdot \pi = 2\pi$$

1. Indique qual dos conjuntos está representado na figura:

$A = \{(x,y,z) \in \mathbb{R}^3 : x + y = z < 1\}$
$B = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 = z < 1\}$
$C = \{(x,y,z) \in \mathbb{R}^3 : \sqrt{x^2+y^2} = z < 1\}$



2. Calcule ou mostre que o limite seguinte não existe: $\lim\limits_{(x,y)\to(0,0)} \cos\left(\dfrac{2\pi x^2}{x^2+y^4}\right)$.

3. Considere a função $f : \mathbb{R}^2 \to \mathbb{R}$ definida por $f(x,y) = (xy)^{1/5}$.
Calcule as derivadas parciais $\dfrac{\partial f}{\partial x}(0,0)$ e $\dfrac{\partial f}{\partial y}(0,0)$. Será que a função f é diferenciável na origem?

3 $\dfrac{\partial f}{\partial x}(0,0) = \dfrac{d\, x^{\frac{1}{5}} y^{\frac{1}{5}}}{dx} = y^{\frac{1}{5}} \cdot \dfrac{1}{5}\, x^{-\frac{4}{5}}$

$f(a+tv) = \left.\dfrac{df(0+t,0)}{dt}\right|_{t=0} = t \times 0 = 0$

$\left.\dfrac{d\, f(t,t)}{dt}\right|_{t=0} \Leftrightarrow t^{2 \times \frac{1}{5}} = \dfrac{d\, t^{\frac{2}{5}}}{dt} = \dfrac{2}{5} \cdot t^{-\frac{3}{5}}$

1) $z = r^2$

2)

Caso
$x = 0$

$\lim\limits_{y \to 0}$ $\cos(0) = 1$

$x = y^2$ $\cos(\pi) = -1$

4. Seja $v = (1,3)$ e $g : \mathbb{R}^2 \to \mathbb{R}$ de classe C^1 tal que $D_v g(2,1) = 4$. Seja $\gamma : \mathbb{R} \to \mathbb{R}^2$ de classe C^1 tal que $\gamma(0) = (2,1)$ e $\gamma'(0) = (1,3)$. Calcule, justificando, $(g \circ \gamma)'(0)$.

5. Considere a função $h : \mathbb{R}^2 \to \mathbb{R}$ definida por $h(x,y) = x^2 + y^2 - 4y - 2x - 1$.

a) Determine os pontos críticos de h e classifique-os.

b) Mostre que h tem um extremo absoluto.

4) $(g \circ \gamma)'(0) = Dg(\gamma(0)) \cdot \gamma'(0) =$

$Dg(2,1) \cdot (1,3) = 4$ //

5) $Dh = \begin{bmatrix} 2x-2 & 2y-4 \end{bmatrix}$ $\begin{cases} 2x-2=0 \\ 2y-4=0 \end{cases} \Leftrightarrow \begin{cases} x=1 \\ y=2 \end{cases}$

$(1,2)$ $h(1,2) = 1+4-8-2-1 = -6$

$H_h = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ ← Def Positiva $(1,2)$ é mínimo

b) Não há pontos em sela logo h tem de ser absoluto logo $(1,2)$ é min abs

[2.0] 6. Calcule $\iint_A f$ em que $f : \mathbb{R}^2 \to \mathbb{R}$ é dada por $f(x, y) = (x-1)(y-1)$ e

$$A = \{(x, y) \in \mathbb{R}^2 : (x-1)^2 + (y-1)^2 < 1 \,;\, 1 < y < x\}$$

usando uma mudança de variáveis adequada.

7. Considere o conjunto $V = \{(x, y, z) \in \mathbb{R}^3 : 0 < y < x < 1 \,;\, y^2 < z < 1\}$. Escreva uma expressão para o volume de V em termos de integrais iterados da forma:

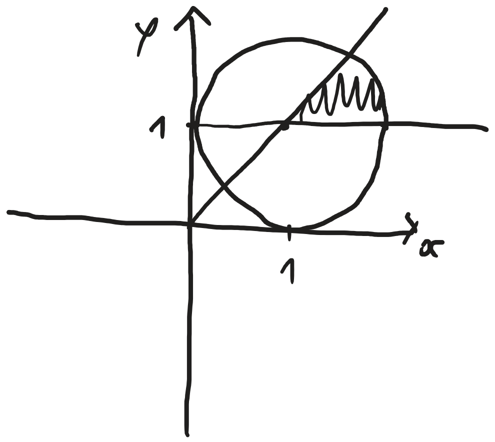
[1.0] a) $\int(\int(\int dz)dx)dy$;

[2.0] b) $\int(\int(\int dy)dx)dz$.

$(x-1)^2+(y-1)^2=\rho^2$

$(x-1)=\rho\cos(\theta)$ $\quad \rho\in[0;1]$

$(y-1)=\rho\sin(\theta)$ $\quad \theta\in[0;\frac{\pi}{4}]$

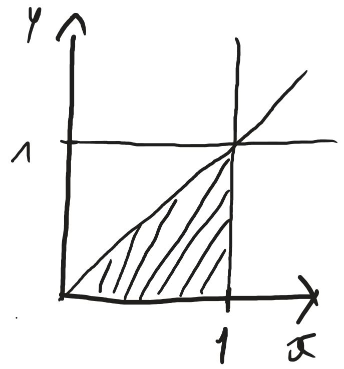


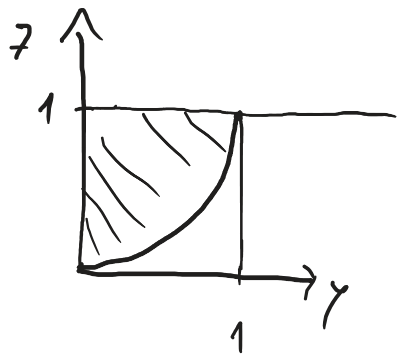
$$\int_0^1\int_0^{\frac{\pi}{4}} \rho^2\cos(\theta)\sin(\theta)\cdot\rho \, d\theta \, d\rho$$

$$\frac{1}{4}\int_0^1 \rho^3 \, d\rho = \frac{1}{4}\left[\frac{\rho^4}{4}\right]_0^1 = \frac{1}{16}$$

$P(fg) = P(f)g - P(P(f)g')$

7a)





$$\int_0^1\int_y^1\int_{y^2}^1 1 \, dz \, dx \, dy$$

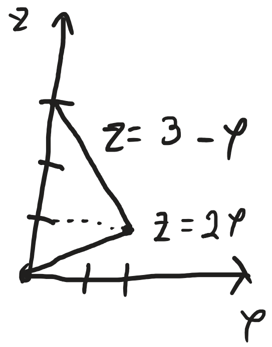
$$\int_0^1 dz \qquad \int_0^{\sqrt{z}}\int_0 \qquad \int_{\sqrt{z}}^1\int_0^{\sqrt{z}} dy$$

$0<y<x<1$

$y<\sqrt{z}$

8. Considere o triângulo plano T definido pelos pontos $(0,0)$, $(1,2)$, $(0,3)$. Calcule o volume do sólido de revolução que resulta da rotação de $180°$ do triângulo T em torno da sua aresta $[(0,0),(0,3)]$.

9. Seja $f : \mathbb{R}^n \to \mathbb{R}$ uma função de classe C^1 tal que $f(0) = 0$. Mostre que existem funções $g_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, 2, \ldots n$, tais que $f(x) = \sum_{i=1}^n x_i g_i(x)$, em que $x = (x_1, x_2, \ldots x_n)$.



$x=\rho\cos(\theta)$ $\quad \theta\in[0,\pi]$

$y=\rho\sin(\theta)$

$z=z$

$\rho^2=x^2+y^2$

$$\int_0^{\pi}\int_0^1\int_{2\rho}^{3-\rho} \rho \, dz \, d\rho \, d\theta =$$

$$\pi\int_0^1\left[\frac{\rho^2}{2}\right]_{2\rho}^{3-\rho} = \pi\int_0^1 \frac{9-6\rho+\rho^2-4\rho^2}{2} \, d\rho = \frac{\pi}{2}\left[9\rho-3\rho^2-\rho^3\right]_0^1 = \frac{5\pi}{2}$$