1º Semestre 13/14

T1 9h

G I

1.) $P(V) = 0.02$ $\qquad P(D) = ?$

a) $P(D|V) = 0.99$

$P(\bar{D}|\bar{V}) = 0.9995 \Rightarrow P(D|\bar{V}) = 0.0005$

$P(D) = P(D|V)\cdot P(V) + P(D|\bar{V})(1-P(V)) = 0.02029$

b) $P(V|D) = \dfrac{P(V\cap D)}{P(D)} = \dfrac{P(D|V)\cdot P(V)}{P(D)} = 0.97585$

2)

a) $X(t) \sim Poisson(\lambda t)$

$E(X(1)) = \lambda = 8$

$V(X(1)) = \lambda = 8$

$X(0.25) \sim Poisson(2)$

$P(X(0.25) \geq 3) = 1 - P(X(0.25) < 3) = 1 - P(X(0.25) \leq 2) = 1 - F_{X(0.25)}(2) = 1 - 0.6767$
$= 0.3233$

b) $Y \sim Bin(n=5, p=0.3233)$

$P(Y \leq 1) = F_Y(1) = [0.4284 ; 0.5282]$

c) $1 - P(X(0.25) = 0) = 1 - F_{X(0.25)}(0)$

G II

1) $X \sim Unif[a,b]$

a)

$$\begin{cases} E(X) = \dfrac{b+a}{2} = 50\,kg \\ P(X>48) = 0.4 \end{cases} \Leftrightarrow \begin{cases} \int_{48}^{b} \dfrac{1}{b-a}\,dx = 0.4 \\ a+b = 100 \end{cases} \Leftrightarrow \begin{cases} 0.4a + 0.1b = 48 \\ a+b = 100 \end{cases} \Leftrightarrow \begin{cases} a = 47.5 \\ b = 52.5 \end{cases}$$

b) Pelo TLC

$E[T] = 100\, E[T] = 5000$

$V[T] = 100\, V[T] = \dfrac{2500}{12}$

$T = \sum_{i=1}^{100} X$

$\dfrac{T - E[T]}{\sqrt{V[T]}} \overset{a}{\sim} N(0,1)$

$P(T > 5050) = P\left(\dfrac{T-5000}{\sqrt{\frac{2500}{12}}} > \dfrac{5050-5000}{\sqrt{\frac{2500}{12}}}\right) \approx 1 - \phi(\sqrt{12})$

2)

X\Y	0	1	2
0	0.2	0.3	0.1
1	0.1	0.2	0.1

a) $E(X|Y=1) = \dfrac{0.3}{0.5}\cdot 0 + \dfrac{0.2}{0.5}\cdot 1 = \dfrac{2}{5}$

b) $Corr(X,Y) = \dfrac{Cov(X,Y)}{\sqrt{V(X)\,V(Y)}}$

$E(X) = 0.6\cdot 0 + 0.4\cdot 1 = 0.4$

$E(X^2) = 0^2\cdot 0.6 + 1^2\cdot 0.4 = 0.4$

$E(Y) = 0\cdot 0.3 + 1\cdot 0.5 + 2\cdot 0.2 = 0.9$

$E(Y^2) = 0^2\cdot 0.3 + 1^2\cdot 0.5 + 2^2\cdot 0.2 = 1.3$

$E(XY) = 0\cdot(0.2+0.3+0.1+0.1) + 1\cdot 0.2 + 2\cdot 0.1 = 0.4$

$Cov(X,Y) = E(XY) - E(X)E(Y)$

$V(X) = E(X^2) - E(X)^2 = 0.24$

$V(Y) = E(Y^2) - E(Y)^2 = 0.49$

$Corr(X,Y) = 0.1166$

1º Semestre 13/14

T2 9h

1)

$X \sim N(0, \theta)$

$\theta = \sigma^2 = Var(X)$

$L(\theta | x_1, \dots, x_n) = \prod_{i=1}^{n} f(x_i, n)$

$L(\hat{\theta} | x_1, \dots, x_n) = \max_{\theta} L(\theta | x_1, \dots, x_n)$

$L(\theta | x_1, \dots, x_n) = \prod_{i=1}^{n} \dfrac{1}{\sqrt{2\pi\theta}} \cdot e^{-\frac{x_i^2}{2\theta}} = \left(\dfrac{1}{\sqrt{2\pi\theta}}\right)^n e^{-\sum_{i=1}^{n} \frac{x_i^2}{2\theta}}$

$\log(L(\theta | x_1, \dots, x_n)) = -\dfrac{n}{2}\log(2\pi\theta) - \sum_{i=1}^{n} \dfrac{x_i^2}{2\theta}$

$\dfrac{d \log(L(\theta | x_1, \dots, x_n))}{d\theta} = -\dfrac{n}{2\theta} + \sum_{i=1}^{n} \dfrac{x_i^2}{2\theta^2} = 0 \Rightarrow \theta = \sum_{i=1}^{n} \dfrac{x_i^2}{n}$

$\left.\dfrac{d^2 \log(L(\theta | x_1, \dots, x_n))}{d\theta^2}\right|_{\theta = \sum_{i=1}^{n} \frac{x_i^2}{n}} = \left.\dfrac{n}{2\theta^2} - \sum_{i=1}^{n} \dfrac{x_i^2}{\theta^3}\right|_{\theta = \sum_{i=1}^{n} \frac{x_i^2}{n}} = -\dfrac{n^3}{2\left(\sum_{i=1}^{n} \frac{x_i^2}{n}\right)} < 0$

Logo $\theta_{MV} = \sum_{i=1}^{n} \dfrac{x_i^2}{n}$

$T = \dfrac{1}{n}\sum_{i=1}^{n} x_i^2$ $\qquad E(T) = E\left(\dfrac{1}{n}\sum_{i=1}^{n} x_i^2\right) = \dfrac{\sum_{i=1}^{n} E(x_i^2)}{n} = E(x^2) = Var(x) + E(x)^2 = \theta$

2)

a) $H_0: p \leq 1$ $\quad X \sim Ber(p)$

$$Z = \frac{\bar{X} - E(X)}{\sqrt{\frac{Var(X)}{n}}} = \frac{\bar{X} - p}{\sqrt{\frac{p(n-p)}{n}}} \overset{a}{\sim} N(0,1)$$

$Z_0 = 0.63564$

Valor-p $= 1 - \Phi(Z_0) = 0.2625 > 0.05$ Logo não se rejeita

1.)

$H_0: X \sim Poisson(5)$

$H_1: X \nsim Poisson(5)$

i	Classe	O_i	p_i^0	$E_i = np_i^0$	
1	$[0,2]$	30	0.1247	24.94	24.94
2	$[3;7]$	140	0.7419	148.38	148.38
3	$[8;10]$	30	0.1197	23.98	$\rightarrow$ 26.68
4	$[11;+\infty[$	0	0.0137	$\boxed{2,74}$	

5<

$$Q_0 = \sum_{i=1}^{3} \frac{(O_i - E_i)^2}{E_i} \underset{H_0}{\overset{a}{\sim}} \chi^2_{(2)}$$

$q_0 = 1.9130$

Valor-p $= 1 - F_{\chi^2_2}(1.9130) \rightarrow]0.3$

2º Semestre 13/14 9h T1

1)

$P(C) = 0.3$

$P(B|C) = 0.4$

$P(\bar{B}|\bar{C}) = 0.8$

$P(C|B) = \frac{P(C\cap B)}{P(B)} = \frac{0.12}{P(B|C)\cdot P(\bar{C})}$

$\frac{P(B\cap C)}{P(C)} = 0.4 \to P(B\cap C) = 0.12$

$1 - P(B|\bar{C}) = 0.8$

$\frac{P(B\cap \bar{C})}{P(\bar{C})} = 0.8 \to P(B\cap\bar{C}) = 0.14$

1) $X \sim N(\mu, \sigma)$

$\mu = 70$

$\sigma = 4$

$P(X>75) = P\left(\frac{X-\mu}{\sigma} > \frac{75-\mu}{\sigma}\right) = 1-\phi\left(\frac{75-\mu}{\sigma}\right) = 1-\phi(1.25) =$

$= 1-0.8944 = 0.1056$

b) $P(X\leq\xi) > 0.75 \wedge P(X\geq\xi) > 0.75$

$P\left(X \geq \frac{\xi-\mu}{\sigma}\right) > 0.75$ $\quad \frac{\xi-\mu}{\sigma} = \phi^{-1}(0.75) \Rightarrow \xi = 72.698$

c) $P(C_3) = 0.1056$

$P(C_1) = P\left(\frac{X-\mu}{\sigma} \leq \frac{65-\mu}{\sigma}\right) = 1-\phi\left(\left|\frac{65-\mu}{\sigma}\right|\right) = 1-\phi(1.25) = 0.1056$

$P(C_2) = 1-0.1056-0.1056 = 0.7888$

$E(L) = 0.1056\cdot 5 + 0.1056\cdot 1 + 0.7888\cdot 3 = 3$

$Var(L) = E(L^2) - E(L)^2 = 0.8448$

$\sigma = \sqrt{Var(L)} = 0.9191$

1.

$L(\theta|x_1,\dots,x_n) = \prod_{i=1}^{n} f(x_i,\theta)$ $\quad n = 10$, $\sum_{i=1}^{10} x_i = 50$

$L(p|\dots) = \prod_{i=1}^{n} p(1-p)^{x_i-1} = p^n(1-p)^{\sum_{i=1}^{n} x_i - n}$

$n\log(p) + \log(1-p)\sum_{i=1}^{n}(x_i-n)$

$\frac{dL}{dp} = \frac{n}{p} - \frac{\sum_{i=1}^{n}(x_i-n)}{1-p} \to \frac{dL}{dp} = 0 \quad p = \frac{n}{\sum x_i}$

2)

$p = 0.2$

$\bar{p} = 0.8$

$X \sim Geo(\bar{p})$

$P(X\geq 3) = 1-P(X\leq 2) = 0.04$

b) $E(X) = 1.25$

$P(X\geq\xi) \geq 0.5 \wedge \underbrace{P(X\leq\xi)}_{0.8} \geq 0.5$

Mediana $\xi = 1$

$1-\overbrace{P(X\leq 1-\xi)}^{0} \geq 0.5$

Logo 1 é mediana

c) $n = 100$

$E(X_i) = 1.25$

$Var(X_i) = \frac{1-p}{p^2} = 0.3125 \to E(Y) = 125$, $Var(Y) = 31.25$

$P(X_i \geq 120) \to 1-P(X_i\leq 120)$

$1-P\left(\frac{X_i - E(Y)}{\sqrt{Var(Y)}} \leq \frac{120-E(Y)}{\sqrt{Var(Y)}}\right) \overset{a}{\sim} N(0,1)$

$1-\phi(-0.89) \to \phi(0.89) =$

2)

Y \ X	0	1
1	$\frac{1}{18}$	$\frac{3}{18}$
2	$\frac{2}{18}$	$\frac{4}{18}$
3	$\frac{3}{18}$	$\frac{5}{18}$

$E(X) = \frac{12}{18} = \frac{2}{3}$

$E(Y) = 1\cdot\frac{4}{18} + 2\cdot\frac{6}{18} + 3\cdot\frac{8}{18} = \frac{40}{18}$

$E(YX) = 1\cdot 1\cdot\frac{3}{18} + 2\cdot 1\cdot\frac{4}{18} + 3\cdot 1\cdot\frac{5}{18} =$

$= \frac{26}{18}$

$Cov(X,Y) = -0.037 \neq 0$ Logo X e Y são dependentes

2.

a) $T = \frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{(n-1)}$

$1-\alpha = 0.9 = P(a<T<b)$

$a: F_{\chi^2_{(30)}}(a) = \frac{0.1}{2} = 0.05 \to a = 18.493$

$b: F_{\chi^2_{(30)}}(b) = 1-\frac{0.1}{2} = 0.95 \to b = 43.773$

$a \leq \frac{(n-1)S^2}{\sigma^2} \leq b \Leftrightarrow$

$I_{0.9}(\sigma^2) = \left[\frac{(n-1)S^2}{b} ; \frac{(n-1)S^2}{a}\right]$

i	O_i	p_i^0	np_i^0	$\frac{(O_i-E_i)^2}{E_i}$
$[0;5]$	33	0.4047	32.53	0.007
$]5;10]$	18	0.2488	17.41	0.020
$]10;20]$	15	0.2044	14.31	0.034
$]20;\infty[$	4	0.0821	5.75	0.593

$\to t_0 = 0.593$

$F_X(x) = \int_{-\infty}^{x} f_X(y)dy = \int_0^x \lambda e^{-\lambda y}dy = 1-e^{-\lambda x}$

$P_1 = F_X(5)$ $\quad P_3 = F_X(20) - F_X(10)$

$P_2 = F_X(10) - F_X(5)$

1) $P(A \cup B \cup C) + P(A \cap (B \cup C)) + P(B \cap C)$

$P(A) + P(B) + P(C) - \cancel{P(A \cap C)} - \cancel{P(B \cap C)} - \cancel{P(A \cap B)} + \cancel{P(A \cap B \cap C)} + \underbrace{P((A \cap B) \cup (A \cap C))}_{} + P(B \cap C)$

$\cancel{P(A \cap B)} + \cancel{P(A \cap C)} - \cancel{P(A \cap B \cap C)} + \cancel{P(B \cap C)}$

$P(A) + P(B) + P(C)$

2.) a)

2) C : Cura

$P(C) = 0.7$ $\quad X \rightarrow$ não curadas

$n = 20$

$X \sim Bin(n=20, p=0{,}3)$

$P(X \geq 5) = 1 - P(X \leq 4) = 1 - 0.2375 = 0.7625$

$P(X \leq 4 | X \geq 1) = \frac{P(0 < X \leq 4)}{1 - P(X=0)} = 0.2336$

3) $P(X=3) = P(X=4)$

$\frac{e^{-\lambda}\lambda^3}{3!} = \frac{e^{-\lambda}\lambda^4}{4!} \Leftrightarrow 4\cancel{e^{-\lambda}}\lambda^3 = \cancel{e^{-\lambda}}\lambda^4 \Leftrightarrow 4\lambda^3 = \lambda^4 \Leftrightarrow$

$\lambda = 4$

$X \sim Poi(\lambda = 4)$

$Y = 10X \sim \{ E(Y) = 40$

$P(Y \geq 50) = 1 - P(Y \leq 49) = 1 - F_{Po(40)}(49) = 0.0703$

1 $X \sim Exp(0.1)$

$E(X) = 10$

a) $P(X > 40 | X < 10) \rightarrow P(X > 30) = e^{-\lambda x} = 0.0498$

b) $P(X \leq \xi) = 0.25$

$1 - e^{-\lambda \xi} = 0.25 \Rightarrow -\lambda\xi = \ln(0.75)$

$\xi = \frac{\ln(0.75)}{-\lambda} = 2.8768$

c) $Y = 50X$

$E(Y) = 50E(X) = 500$

$V(Y) = 50V(X) = 5000$

$T = \frac{Y - E(Y)}{\sqrt{V(Y)}} \overset{a}{\sim} N(0,1)$

$P(Y \geq 300) = 1 - P\left(T < \frac{300 - E(Y)}{\sqrt{V(Y)}}\right) = \phi(2.8284) \approx 0.9977$

1.

	o	p_i	E_i	$\frac{(O-E_i)^2}{E_i}$
AA	65	0.25	71	0.5070
AB	152	0.50	142	0.7042
BB	67	0.25	71	0.2254

$\rightarrow t_0 = 1.4366$

$P = 1 - F_{\chi^2_{(2)}}(1.4366) =$
$= 0.4876$

2)

X \ Y	0	1	2
0	0.2	0.0	0.1
1	0.2	0.0	0.1
2	0.0	0.3	0.1

$P(Y=0 | X=0) = \frac{2}{3}$

$\frac{a}{a+b+0.1} = \frac{2}{3}$

$a + b = 0.2$

$a = 0.2$

$b = 0.0$

$E(X | Y=2) = 1 \cdot \frac{1}{3} + 2 \cdot \frac{1}{3} = 1$

$V(X|Y=2) = \frac{5}{3} - 1^2 = \frac{2}{3}$

$E((X|Y=2)^2) = 1 \cdot \frac{1}{3} + 4 \cdot \frac{1}{3} = \frac{5}{3}$

$P(X \geq \xi | Y=2) \geq 0.5 \wedge P(X \leq \xi | Y=2) \geq 0.5$

Testar $\xi = 1$: $\quad \frac{2}{3} \geq 0.5 \wedge \frac{2}{3} \geq 0.5$

$L(p | x_1, \ldots, x_n) = \prod_{i=1}^{n} f(x_i, p) = \prod_{i=1}^{n} p^{x_i}(1-p)^{1-x_i}$

$\log L(p | x_1, \ldots, x_n) = \log(p)\sum_{i=1}^{n} x_i + \log(1-p)\sum_{i=1}^{n}(1 - x_i)$

$\frac{d \log L}{dp} = \frac{\sum_{i=1}^{n} x_i}{p} - \frac{n - \sum_{i=1}^{n} x_i}{1-p} = 0$

$(1-p)\sum_{i=1}^{n} x_i - p\left(n - \sum_{i=1}^{n} x_i\right) = 0$

$\sum_{i=1}^{n} x_i - p\sum_{i=1}^{n} x_i - p\left(n - \sum_{i=1}^{n} x_i\right) = 0$

$p = \frac{\sum_{i=1}^{n} x_i}{\sum_{i=1}^{n} x_i - \sum_{i=1}^{n} x_i + n} = \bar{x}$

2) $\mu = 19 \qquad \mu \leq 16$ e $\mu > 16$

$n = 40$

$\sigma = 5$

b) $\alpha = 0.01$

$Z = \frac{\bar{X} - p}{\sqrt{\frac{\bar{x}(1-\bar{x})}{n}}} \overset{a}{\sim} N(0,1)$ $\qquad \phi(a) = \frac{1+\gamma}{2} \rightarrow a = \phi^{-1}\left(\frac{1+\gamma}{2}\right)$

$T = \frac{\bar{X} - \mu}{\sigma / \sqrt{n}} \sim N(0,1)$

$P\left(\frac{\bar{x} - 16}{\sigma/\sqrt{n}} > 1.6449\right)$

$P(-a \leq Z \leq a) = \gamma \Leftrightarrow P\left(-a \leq \frac{\bar{X} - p}{\sqrt{\frac{\bar{x}(1-\bar{x})}{n}}} \leq a\right) = \gamma$

$t_0 = \frac{19 - 16}{5/\sqrt{40}} = \frac{3\sqrt{40}}{5} = 3.794$

$P\left(-a\sqrt{\frac{\bar{x}(1-\bar{x})}{n}} - \bar{x} \leq -p \leq a\sqrt{\frac{\bar{x}(1-\bar{x})}{n}} - \bar{x}\right) = \gamma$ $\qquad P\left(\bar{x} - a\sqrt{\frac{\bar{x}(1-\bar{x})}{n}} \leq p \leq \bar{x} + a\sqrt{\frac{\bar{x}(1-\bar{x})}{n}}\right) = \gamma$