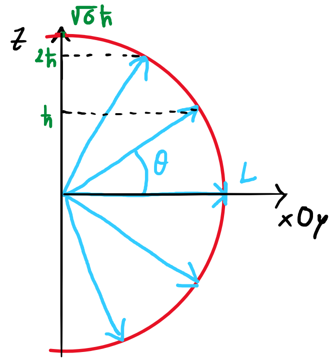
Calculate all possible values of the angular momentum L, of its projection on the z-axis L_z, and of the magnetic dipole momentum along z μ_z of an electron when l = 2, and plot them graphically.

$L^2=\hbar^2 l(l+1)$, $l=2 \rightsquigarrow L^2=6\hbar^2 \Rightarrow L=\pm\sqrt{6}\,\hbar$

$L_z=\hbar m_l$, $m_l=\{0,\pm1,\pm2\}$ $\quad L_z=\{0,\pm\hbar;\pm2\hbar\}$

$\mu_z=\dfrac{q}{2m_e}L_z=\left\{0;\pm\dfrac{q\hbar}{2m_e};\pm\dfrac{q\hbar}{m_e}\right\}$

$\sin(\theta)=\dfrac{L_z^1}{L} \Rightarrow \theta=\arcsin\left(\dfrac{L_z^1}{L}\right)=24.1^\circ$



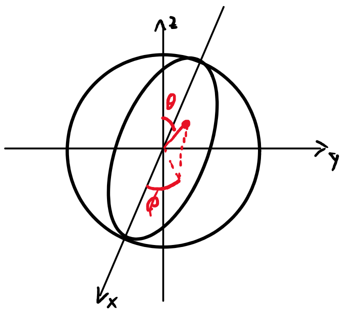
$\hbar=1.055\times10^{-34}\,Js$

$q=1.602\times10^{-19}\,C$

$m_e=9.109\times10^{-31}\,Kg$

A qubit is localized on the Bloch sphere by angles $\theta=50^\circ$ and $\phi=10^\circ$. Draw the state on the Bloch sphere and calculate the state vector in the $\{|0\rangle,|1\rangle\}$ basis and the probability of measuring the basis states.

$\theta=50^\circ \quad \phi=10^\circ$



$|\psi\rangle=\cos(\frac{\theta}{2})|0\rangle+\sin(\frac{\theta}{2})e^{i\phi}|1\rangle$

$|\psi\rangle=\begin{bmatrix}\alpha\\ \beta\end{bmatrix}$

$\alpha=\cos(\frac{\theta}{2})=0.906 \Rightarrow P_0(|\psi\rangle)=|\langle\psi|0\rangle|^2=0.821=82.1\%$

$\beta=\sin(\frac{\theta}{2})e^{i\phi}=0.416+0.073i \Rightarrow P_1(|\psi\rangle)=|\langle\psi|1\rangle|^2=0.179=17.9\%$

Consider a qubit $|\psi\rangle=\left(\frac{1}{2}+\frac{i}{2}\right)|0\rangle-\left(\frac{1}{2\sqrt{2}}+i\frac{\sqrt{3}}{2\sqrt{2}}\right)|1\rangle$.

a. Locate the state on the Bloch sphere by calculating the corresponding angles θ,ϕ.
b. Calculate the global rotation angle δ and the equivalent state $|\psi'\rangle$ with purely real α' coefficient.

a) $\alpha=\frac{1}{2}+\frac{i}{2}=\cos(\frac{\theta}{2})e^{i\gamma}$

$|\alpha|=\frac{\sqrt{2}}{2} \Rightarrow \theta=2\arccos(\frac{\sqrt{2}}{2})=\frac{\pi}{2}$

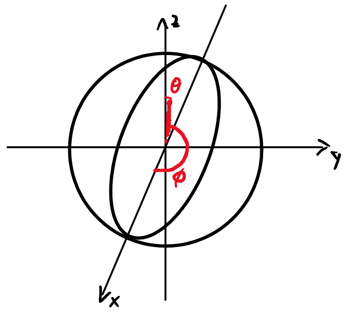
$\sphericalangle\alpha=\gamma=\arctan(1)=\frac{\pi}{4}$

$\beta=\sin(\frac{\theta}{2})e^{i\gamma}e^{i\phi}$

$-\left(\frac{1}{2\sqrt{2}}+i\frac{\sqrt{3}}{2\sqrt{2}}\right)=\beta \rightarrow$ 3th Quadrant

$\sphericalangle\beta=\gamma+\phi \Leftrightarrow \arctan(\sqrt{3})-\gamma=\phi \Leftrightarrow \phi=\frac{\pi}{3}-\frac{\pi}{4}=\frac{\pi}{12} \Rightarrow \frac{13\pi}{12}$

$|\psi\rangle=e^{i\frac{\pi}{4}}\left(\cos(\frac{\pi}{4})|0\rangle+\sin(\frac{\pi}{4})e^{i\frac{13\pi}{12}}|1\rangle\right)$



b) $\sphericalangle\alpha=0 \Rightarrow \delta+\gamma=0 \Leftrightarrow \delta=-\frac{\pi}{4} \Rightarrow |\psi'\rangle=\cos(\frac{\pi}{4})|0\rangle+\sin(\frac{\pi}{4})e^{i\frac{13\pi}{12}}|1\rangle=$

$=\frac{\sqrt{2}}{2}|0\rangle-(0.683+0.183i)|1\rangle$

Consider two states $|\psi\rangle$, $|\psi'\rangle$ differing only by a global phase factor $e^{i\gamma}$. Show that the probability of measuring a state $|s\rangle$ is identical for $|\psi\rangle$ and $|\psi'\rangle$, for any target state $|s\rangle$.

$$|\psi\rangle e^{i\gamma} = |\psi'\rangle \qquad P_s(|\psi\rangle) = |\langle\psi|s\rangle|^2 = |\alpha^*\langle 0|s\rangle + \beta^*\langle 1|s\rangle|^2$$

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \qquad P_s(|\psi'\rangle) = |\langle\psi'|s\rangle|^2 = \underbrace{|e^{-i\gamma}|^2}_{1} \cdot |\alpha^*\langle 0|s\rangle + \beta^*\langle 1|s\rangle|^2 = P_s(|\psi\rangle)\,!$$

Consider the Stern-Gerlach experimental setup in Fig. 1, where the input qubit is prepared in state $|\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle - \left(\frac{1+\sqrt{3}}{4} - i\frac{1-\sqrt{3}}{4}\right)|1\rangle$. Calculate the measurement probability for states $|x_+\rangle$, $|x_-\rangle$ and $|y_+\rangle$, $|y_-\rangle$ after the corresponding experimental setups. Suppose then to collimate the $|x_+\rangle$, $|x_-\rangle$ output beams. Calculate the measurement probability for $|y_+\rangle$, $|y_-\rangle$ after the second SG setup.

$$|x_+\rangle = \tfrac{1}{\sqrt{2}}|0\rangle + \tfrac{1}{\sqrt{2}}|1\rangle$$
$$|x_-\rangle = \tfrac{1}{\sqrt{2}}|0\rangle - \tfrac{1}{\sqrt{2}}|1\rangle$$
$$|y_+\rangle = \tfrac{1}{\sqrt{2}}|0\rangle + \tfrac{i}{\sqrt{2}}|1\rangle$$
$$|y_-\rangle = \tfrac{1}{\sqrt{2}}|0\rangle - \tfrac{i}{\sqrt{2}}|1\rangle$$

$$|\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle - \left(\frac{1+\sqrt{3}}{4} - i\frac{1-\sqrt{3}}{4}\right)|1\rangle = \cos\left(\frac{\pi}{4}\right)|0\rangle + \sin\left(\frac{\pi}{4}\right)e^{-i\frac{11}{12}\pi}$$

$$P_{x_+}(|\psi\rangle) = |\langle\psi|x_+\rangle|^2 = \left|\frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}}\langle 0|0\rangle - \frac{1}{\sqrt{2}}\cdot\left(\frac{1+\sqrt{3}}{4} + i\frac{1-\sqrt{3}}{4}\right)\langle 1|1\rangle\right|^2 = \left|\frac{1}{2} - \frac{1+\sqrt{3}}{4\sqrt{2}} - i\frac{1-\sqrt{3}}{4\sqrt{2}}\right|^2 = *$$

$$* = |\langle\psi|x_+\rangle|^2 = 0.017 = 1.7\%$$

$$P_{x_-}(|\psi\rangle) = |\langle\psi|x_-\rangle|^2 = \left|\tfrac{1}{\sqrt{2}}\cdot\tfrac{1}{\sqrt{2}}\langle 0|0\rangle + \frac{1}{\sqrt{2}}\cdot\left(\frac{1+\sqrt{3}}{4} + i\frac{1-\sqrt{3}}{4}\right)\langle 1|1\rangle\right|^2 = 1 - |\langle\psi|x_+\rangle|^2 = 0.983 = 98.3\%$$

$$P_{y_+}(|x_+\rangle) = |\langle x_+|y_+\rangle|^2 = \left|\tfrac{1}{2}\langle 0|0\rangle + \tfrac{i}{2}\langle 1|1\rangle\right|^2 = 0.5 \Rightarrow 50\% \quad \Rightarrow P_{y_+}(|\psi\rangle) = P_{y_+}(|x_+\rangle)\cdot P_{x_+}(|\psi\rangle) = 0.85\%$$

$$P_{y_-}(|x_+\rangle) = |\langle x_+|y_-\rangle|^2 = \left|\tfrac{1}{2}\langle 0|0\rangle - \tfrac{i}{2}\langle 1|1\rangle\right|^2 = 0.5 \Rightarrow 50\% \quad \Rightarrow P_{y_-}(|\psi\rangle) = P_{y_-}(|x_+\rangle)\cdot P_{x_+}(|\psi\rangle) = 0.85\%$$

Collimated $\Rightarrow$ Use a mixed ensemble $\rho = \sum_i P_i\,|\psi_i\rangle\langle\psi_i|$

$$\rho = 0.017|x_+\rangle\langle x_+| + 0.983|x_-\rangle\langle x_-| = 0.017\cdot\tfrac{1}{2}\begin{bmatrix}1\\1\end{bmatrix}\begin{bmatrix}1 & 1\end{bmatrix} + 0.983\cdot\tfrac{1}{2}\begin{bmatrix}1\\-1\end{bmatrix}\begin{bmatrix}1 & -1\end{bmatrix} =$$

$$= \frac{0.017}{2}\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix} + \frac{0.983}{2}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix} = \begin{bmatrix}0.5 & -0.483\\-0.483 & 0.5\end{bmatrix}$$

$$P_{y_+} = |y_+\rangle\langle y_+| = \frac{1}{2}\begin{bmatrix}1\\i\end{bmatrix}\begin{bmatrix}1 & -i\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1 & -i\\i & 1\end{bmatrix}$$

$$P_{y_-} = |y_-\rangle\langle y_-| = \frac{1}{2}\begin{bmatrix}1\\-i\end{bmatrix}\begin{bmatrix}1 & i\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1 & i\\-i & 1\end{bmatrix}$$

$$\rho' = P_{y_+}\cdot\rho = \frac{1}{2}\begin{bmatrix}1 & -i\\i & 1\end{bmatrix}\begin{bmatrix}0.5 & -0.483\\-0.483 & 0.5\end{bmatrix} = \begin{bmatrix}0.5 + i\,0.483 & -0.483 - i\,0.5\\-0.483 + i\,0.5 & 0.5 - i\,0.483\end{bmatrix}$$

$$P_{y+}(\rho) = Tr(P_{y_+}\cdot\rho) = Tr(\rho') = \frac{1}{2}$$

$$P_{y-}(\rho) = 1 - P_{y+}(\rho) = 0.5 = \tfrac{1}{2}$$

Derive the Pauli operator for direction $\vec{n}$ described by $\theta = \frac{\pi}{2}$, $\phi = \frac{\pi}{4}$, calculate its corresponding eigenvectors and eigenvalues, and plot them on the Bloch sphere.

$$\hat{\sigma}_m = m_x \hat{X} + m_y \hat{Y} + m_z \hat{Z} \Rightarrow \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1-i \\ 1+i & 0 \end{bmatrix}$$

$$m_x = \sin(\theta) \cos(\phi) = \frac{1}{\sqrt{2}}$$

$$m_y = \sin(\theta) \sin(\phi) = \frac{1}{\sqrt{2}}$$

$$m_z = \cos(\theta) = 0$$

or

$$\hat{\sigma}_m = \begin{bmatrix} \cos(\theta) & \sin(\theta) e^{-i\phi} \\ \sin(\theta) e^{i\phi} & \cos(\theta) \end{bmatrix} = \begin{bmatrix} 0 & e^{-i\pi/4} \\ e^{i\pi/4} & 0 \end{bmatrix}$$

Eigenvalues

Eigenvector => states where you can collapse on.

$$\det \begin{vmatrix} \lambda & e^{-i\pi/4} \\ e^{i\pi/4} & \lambda \end{vmatrix} = \lambda^2 - 1 \Rightarrow \lambda = \pm 1$$

$$\hat{\sigma}_m v = \lambda v \Rightarrow \begin{bmatrix} 0 & e^{-i\frac{\pi}{4}} \\ e^{i\frac{\pi}{4}} & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \lambda \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$\begin{cases} e^{-i\frac{\pi}{4}} v_1 = \lambda v_2 \\ e^{i\frac{\pi}{4}} v_2 = \lambda v_1 \end{cases} \Rightarrow \lambda = 1 \rightarrow v = \begin{bmatrix} 1/\sqrt{2} \\ 1/2(1+i) \end{bmatrix}$$

$$\lambda = -1 \rightarrow v = \begin{bmatrix} 1/2\,(1-i) \\ -1/\sqrt{2} \end{bmatrix}$$

$|v_1|^2 + |v_2|^2 = 1 \leftarrow$ Normalization Condition