$P_0(|\psi\rangle) - P_1(|\psi\rangle) = \cos(\theta)$ $P_0(H|\psi\rangle) - P_1(H|\psi\rangle) = \sin(\theta)\cos(\phi)$ $P_0(HS^\dagger|\psi\rangle) - P_1(HS^\dagger|\psi\rangle) = \sin(\theta)\sin(\phi)$

Exercise 1

Consider a state $|\psi\rangle$. Knowing that $P_0(|\psi\rangle) = 0.933, P_1(|\psi\rangle) = 0.067, P_0(H|\psi\rangle) = 0.625,$ $P_1(H|\psi\rangle) = 0.375, P_0(HS^\dagger|\psi\rangle) = 0.7165, P_1(HS^\dagger|\psi\rangle) = 0.2835,$ estimate the angles θ, ϕ localizing the state on the Bloch sphere.

$\nearrow |\langle 0|\psi\rangle|^2$ $\quad \nearrow |\langle +|\psi\rangle|^2$ $\quad \nearrow |\langle y^+|\psi\rangle|^2$

$P_0(|\psi\rangle) = 0.933$ $\quad P_0(H|\psi\rangle) = 0.625$ $\quad P_0(HS^\dagger|\psi\rangle) = 0.7165$

$P_1(|\psi\rangle) = 0.067$ $\quad P_1(H|\psi\rangle) = 0.375$ $\quad P_1(HS^\dagger|\psi\rangle) = 0.2835$

$$H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \quad S^\dagger = \begin{bmatrix} 1 & 0 \\ 0 & -i \end{bmatrix} \quad HS^\dagger = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -i \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}$$

$$H|\psi\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} \alpha+\beta \\ \alpha-\beta \end{bmatrix} \quad HS^\dagger|\psi\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} \alpha-i\beta \\ \alpha+i\beta \end{bmatrix}$$

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + \sin\left(\frac{\theta}{2}\right)e^{i\phi}|1\rangle$$

$|\alpha|^2 = 0.933$ $\quad \frac{1}{2}|\alpha+\beta|^2 = 0.625$ $\quad \frac{1}{2}|\alpha - i\beta|^2 = 0.7165$

$|\beta|^2 = 0.067$ $\quad \frac{1}{2}|\alpha-\beta|^2 = 0.375$ $\quad \frac{1}{2}|\alpha+i\beta|^2 = 0.2835$

$$|\cos\left(\frac{\theta}{2}\right)|^2 = 0.933$$

$$\hookrightarrow \theta = 2\cos^{-1}\left(\sqrt{0.933}\right) = 30^\circ$$

$$\frac{1}{2}\left|\left(\cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right)e^{i\phi}\right)\right|^2 = 0.625 \Leftrightarrow \frac{1}{2}\left(\cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right)e^{i\phi}\right)\left(\cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right)e^{-i\phi}\right) = 0.625$$

$$\cos^2\left(\frac{\theta}{2}\right) + \sin^2\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right)e^{i\phi}\cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right)e^{-i\phi}\cos\left(\frac{\theta}{2}\right) = 1.25$$

$$1 + \sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)\left(e^{i\phi} + e^{-i\phi}\right) = 1.25 \Rightarrow e^{i\phi} + e^{-i\phi} = 1$$

$$\frac{1}{2}\left|\cos\left(\frac{\theta}{2}\right) + e^{-i\frac{\pi}{2}}\sin\left(\frac{\theta}{2}\right)e^{i\phi}\right|^2 = 0.7165 \Leftrightarrow \left(\cos\left(\frac{\theta}{2}\right) + e^{i\frac{\pi}{2}}\sin\left(\frac{\theta}{2}\right)e^{-i\phi}\right)\left(\cos\left(\frac{\theta}{2}\right) + e^{-i\frac{\pi}{2}}\sin\left(\frac{\theta}{2}\right)e^{i\phi}\right) = 1.433$$

$$\cos^2\left(\frac{\theta}{2}\right) + \sin^2\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)e^{i(\phi-\frac{\pi}{2})} + \sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)e^{-i(\phi-\frac{\pi}{2})} = 1.433$$

$$1 + \sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)\left(e^{i(\phi-\frac{\pi}{2})} + e^{-i(\phi-\frac{\pi}{2})}\right) = 1.433 \Rightarrow e^{i(\phi-\frac{\pi}{2})} + e^{-i(\phi-\frac{\pi}{2})} = 1.732$$

$$e^{i\phi} + e^{-i\phi} = 1 \Rightarrow 2\cos(\phi) = 1 \Rightarrow \phi = \pm\frac{\pi}{3}$$

$$e^{i(\phi-\frac{\pi}{2})} + e^{-i(\phi-\frac{\pi}{2})} = 1.732 \quad 2\cos\left(\phi - \frac{\pi}{2}\right) = 1.732 \Rightarrow \phi = \frac{\pi}{3} \Rightarrow \phi = 60^\circ$$

Exercise 3

Consider a state $|\psi\rangle$. Knowing that $P_0(|\psi\rangle) = 0.2, P_1(|\psi\rangle) = 0.8, P_0(H|\psi\rangle) = 0.6,$ $P_1(H|\psi\rangle) = 0.4, P_0(HS^\dagger|\psi\rangle) = 0.7, P_1(HS^\dagger|\psi\rangle) = 0.3,$ estimate the angles θ, ϕ localizing the state on the Bloch sphere.

$P_0(|\psi\rangle) - P_1(|\psi\rangle) = -0.6 = \cos(\theta) \Rightarrow \theta = 2.21$ rad/s

$P_0(H|\psi\rangle) - P_1(H|\psi\rangle) = 0.2 = \sin(\theta)\cos(\phi) \rightarrow \frac{1}{4} = \cos(\phi)$

$P_0(HS^\dagger|\psi\rangle) - P_1(HS^\dagger|\psi\rangle) = 0.4 = \sin(\theta)\sin(\phi) \rightarrow \frac{1}{2} = \sin(\phi)$

$\rangle \tan(\phi) = \frac{1/2}{1/4} = 2$

$\hookrightarrow 1.107$ rad/s

Exercise 2

Consider the truth table:

Input state $\vert\psi_{in}\rangle$	Output state $\vert\psi_{out}\rangle$
$\vert 0\rangle$	$\vert -\rangle$
$\vert 1\rangle$	$\vert +\rangle$

Calculate the operator $\hat{O}$ such that $|\psi_{out}\rangle = \hat{O}|\psi_{in}\rangle$.

$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}a & c\\ b & d\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}a\\b\end{pmatrix}$

$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}a & c\\ b & d\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}c\\d\end{pmatrix}$

$\hat{O} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}$ //

Exercise 3

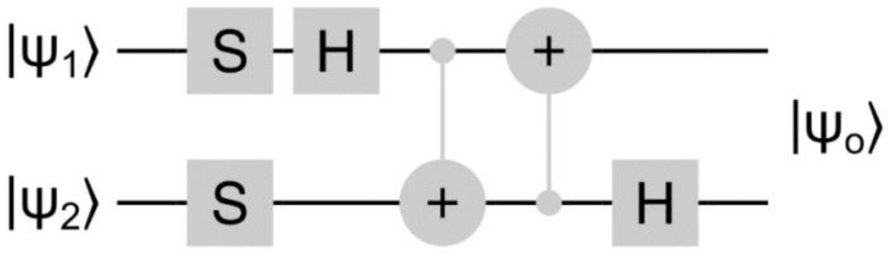
Consider a two-qubit system with state $|\psi\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$. Determine whether the state is a *product state* or *entangled state*.

Product: $|\psi\rangle = |\psi_1\rangle \cdot |\psi_2\rangle = \alpha_1\alpha_2|00\rangle + \alpha_1\beta_2|01\rangle + \beta_1\alpha_2|10\rangle + \beta_1\beta_2|11\rangle$

In this case: $\alpha_1\alpha_2 = 0$, $\beta_1\beta_2 = 0$ $\rangle\!\!\Rightarrow$ at least one α or β is 0 $\Rightarrow$ $\alpha_1\beta_2$, $\beta_1\alpha_2$ $\nwarrow$ one should be zero $\Rightarrow$ Entangled state

Exercise 5

Consider the two-qubit circuit in Fig. 1. Determine the truth table and the corresponding equivalent operator $\hat{O}$.



$$|\psi_o\rangle = (I \times H) \cdot ICNOT \cdot CNOT \cdot (H \times I) \cdot (S \times S) \begin{pmatrix} |\psi_1\rangle \\ |\psi_2\rangle \end{pmatrix}$$

$$S \times S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$H \times I = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$$

$$CNOT = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$ICNOT = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$(I \times H) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$$A = (H \times I) \cdot (S \times S) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & i & 0 \\ 0 & i & 0 & -1 \\ 1 & 0 & -i & 0 \\ 0 & i & 0 & 1 \end{bmatrix}$$

$$B = CNOT \cdot A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & i & 0 \\ 0 & i & 0 & -1 \\ 1 & 0 & -i & 0 \\ 0 & i & 0 & 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & i & 0 \\ 0 & i & 0 & -1 \\ 0 & i & 0 & 1 \\ 1 & 0 & -i & 0 \end{bmatrix}$$

$$C = ICNOT \cdot B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & i & 0 \\ 0 & i & 0 & -1 \\ 0 & i & 0 & 1 \\ 1 & 0 & -i & 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & i & 0 \\ 1 & 0 & -i & 0 \\ 0 & i & 0 & 1 \\ 0 & i & 0 & -1 \end{bmatrix}$$

$$\hat{O} = I \times H \cdot C = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \cdot \frac{1}{\sqrt{2}} \cdot \begin{bmatrix} 1 & 0 & i & 0 \\ 1 & 0 & -i & 0 \\ 0 & i & 0 & 1 \\ 0 & i & 0 & -1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 2i & 0 \\ 0 & 2i & 0 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$|00\rangle \Rightarrow |00\rangle$

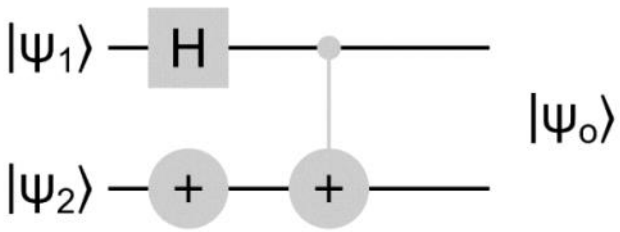
$|01\rangle \Rightarrow i|10\rangle$

$|10\rangle \Rightarrow i|01\rangle$

$|11\rangle \Rightarrow |11\rangle$

Exercise 4

Consider the two-qubit circuit in Fig. 1, where input qubits $|\psi_1\rangle, |\psi_2\rangle$ are prepared in the $|0\rangle$ and $|1\rangle$ state respectively. Determine the output state $|\psi_o\rangle$ of the circuit.



$$|\psi_2\rangle \otimes |\psi_1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad |\overset{T}{\psi_2}\,\overset{C}{\psi_1}\rangle$$

$$+ \otimes H \Rightarrow \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \Rightarrow \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}$$

$$\text{CNOT} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\hat{O} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$$\hat{O} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$|\psi_1\rangle \otimes |\psi_2\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad |\overset{C}{\psi_1}\,\psi_2\rangle$$

$$H \otimes + \rightarrow \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \Rightarrow \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{bmatrix}$$

$$\text{CNOT} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\hat{O} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$$

$$\hat{O} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$