Consider an operator $\hat{O}$ with truth table:

Input state $\vert\psi_{in}\rangle$	Output state $\vert\psi_{out}\rangle$
$\vert 0\rangle$	$\vert -\rangle$
$\vert 1\rangle$	$\vert +\rangle$

Calculate the angles ϕ, θ, λ to implement $\hat{O}$ with two rotations around $\vec{z}$ and one rotation around $\vec{y}$.

$$\hat{O} = \hat{R}_z(\phi) \cdot \hat{R}_y(\theta) \cdot \hat{R}_z(\lambda)$$

$$\hat{R}_z(\delta) = \begin{bmatrix} e^{-i\delta/2} & 0 \\ 0 & e^{i\delta/2} \end{bmatrix} \qquad \hat{R}_y(\delta) = \begin{bmatrix} \cos(\delta/2) & -\sin(\delta/2) \\ \sin(\delta/2) & \cos(\delta/2) \end{bmatrix}$$

$$\hat{O} = |-\rangle\langle 0| + |+\rangle\langle 1| = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\langle 0| + \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\langle 1| =$$

$$= \frac{1}{\sqrt{2}}\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$\hat{O} = \begin{bmatrix} e^{-i(\frac{\phi}{2}+\frac{\lambda}{2})}\cos(\frac{\theta}{2}) & -e^{-i(\frac{\phi}{2}-\frac{\lambda}{2})}\sin(\frac{\theta}{2}) \\ e^{i(\frac{\phi}{2}-\frac{\lambda}{2})}\sin(\frac{\theta}{2}) & e^{i(\frac{\phi}{2}+\frac{\lambda}{2})}\cos(\frac{\theta}{2}) \end{bmatrix}$$

with global phase $e^{i\delta} = e^{-i(\frac{\phi}{2}+\frac{\lambda}{2})}$

$$\hat{O} = \begin{bmatrix} \cos(\frac{\theta}{2}) & -e^{i\lambda}\sin(\frac{\theta}{2}) \\ e^{i\phi}\sin(\frac{\theta}{2}) & \cos(\frac{\theta}{2}) \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$\boxed{\theta = \frac{\pi}{2} \quad \lambda = \pi \quad \phi = \pi}$$

$\delta = -\pi$!

with global phase $\delta = 0$

$$e^{-i(\frac{\phi}{2}+\frac{\lambda}{2})}\cos(\frac{\theta}{2}) = 1$$

$$e^{i(\frac{\phi}{2}+\frac{\lambda}{2})}\cos(\frac{\theta}{2}) = 1 \quad \Rightarrow \phi = \lambda = 0$$

$\phi = 0$

$\lambda = 0$

$\theta = -\frac{\pi}{2}$

$$\begin{bmatrix} \cos(\frac{\theta}{2}) & -\sin(\frac{\theta}{2}) \\ \sin(\frac{\theta}{2}) & \cos(\frac{\theta}{2}) \end{bmatrix} \Leftrightarrow \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \Rightarrow \theta = -\frac{\pi}{2}$$

Exercise 2

Consider an electron subject to a magnetic field $B = 2\,T$ directed along $\vec{z}$. Determine the position of the energy levels $|0\rangle$, $|1\rangle$, the qubit frequency ω_0 and the maximum operation temperature.

Exercise 3

Consider an electron subject to a magnetic field $B = 2\,T$ directed along $\vec{z}$. Determine the Larmor frequency ω_L and the time t_1 for which the magnetic field must be applied to operate a gate $\hat{S}$.

2/3 $\quad B = 2T$

$$\hat{H} = -\vec{\mu}\cdot\vec{B} = -\gamma\vec{S}\cdot\vec{B} = -\gamma\vec{S}_z \cdot B = -\gamma B\frac{\hbar}{2}\hat{\sigma}_z = -\gamma B\frac{\hbar}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \mu_B B\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad \mu_B = -\gamma\frac{\hbar}{2}$$

Energy Level:

$$\hat{H}|0\rangle = \mu_B B\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \mu_B B|0\rangle \Rightarrow E_0 = \mu_B B = 116\,eV$$

$$\hat{H}|1\rangle = \mu_B B\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = -\mu_B B|1\rangle \Rightarrow E_1 = -\mu_B B_0 = -116\,eV$$

$$\hat{S} = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} = \begin{bmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & e^{i\theta(t)} \end{bmatrix}$$

$$\theta(t) = \omega_L t = \frac{\pi}{2}$$

$$\hookrightarrow t_1 = \frac{\pi}{2\omega_L} = 4.46\,ps$$

Frequency

$$\omega_0 = \omega_L = \frac{\Delta E}{\hbar} = 351 \times 10^9\,rad/s = 56\,GHz$$

Max Temp

$$T \ll \frac{\Delta E}{k} \Rightarrow T \ll 2.68\,K \qquad T_{Max} = \frac{T}{1000} = 2.68\,mK\,!$$

Exercise 4

An electron is prepared in a state $|\psi_0\rangle$ with $\theta = 45°$, $\phi = 0°$ and immersed in a static magnetic field $B = 1\,T$ directed along $\vec{z}$. Draw a quoted plot of the probability of measuring the $|0\rangle$ and $|+\rangle$ state along time.

$$|\psi_0\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|1\rangle = 0.924|0\rangle + 0.383|1\rangle$$

$B = 1\,T$

$$\hat{R}_z = \begin{bmatrix} e^{-i\frac{\omega_L t}{2}} & 0 \\ 0 & e^{i\frac{\omega_L t}{2}} \end{bmatrix}$$

$$|\psi(t)\rangle = \hat{R}_z \cdot |\psi_0\rangle = 0.924\, e^{-i\frac{\omega_L t}{2}}|0\rangle + 0.383\, e^{i\frac{\omega_L t}{2}}|1\rangle$$

$$P_{|0\rangle} = |\langle 0|\psi(t)\rangle|^2 = |0.924\, e^{-i\frac{\omega_L t}{2}}|^2 = 0.924^2 = 0.8538$$

$$P_{|+\rangle} = |\langle +|\psi(t)\rangle|^2 = \left|\frac{1}{\sqrt{2}}\left(\langle 0|0\rangle\, 0.924 \cdot e^{-i\frac{\omega_L t}{2}} + \langle 1|1\rangle\, 0.383 \cdot e^{i\frac{\omega_L t}{2}}\right)\right|^2 =$$

$$= \frac{1}{2}|0.924 + 0.383\, e^{i\omega_L t}|^2$$ depends on t!

Exercise 5

An electron is immersed in a static magnetic field $B = 1\,T$ directed along $\vec{z}$. Calculate the timing accuracy of a control system to provide a rotation angle accuracy $\Delta\theta = \frac{\pi}{1000}$.

$B = 1\,T$

$\Delta\theta = \frac{\pi}{1000}$

$$\omega_0 t = \theta \Leftrightarrow \Delta t = \frac{\Delta\theta}{\omega_0} = \frac{\pi}{1000 \cdot 56\,GHz} = 8.9\,fs$$